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Ul’yanov and Nikol’skii-type inequalities

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Abstract

Ul’yanov-type inequalities are extended to include many measures of smoothness. Many of the results are valid for $L_p,\ p>0$. Observations and extensions for needed Nikolskii-type inequalities are given.

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1. Introduction

For a periodic function $L_p(\boldsymbol{T})$ Ul’yanov [Ul] proved the now classical inequalities

$$\omega(f,t)_q \leqslant C\left\{\int_0^t \left(u^{-\theta}\omega(f,u)_p\right)^q \frac{du}{u}\right\}^{1/q} \tag{1.1}$$

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and

$$\|f\|_q \leqslant C \left\{ \left[\int_0^1 \left(u^{-\theta} \omega(f,u)_p \right)^q \frac{du}{u} \right]^{1/q} + \|f\|_p \right\}, \tag{1.2}$$

where $\theta = \frac{1}{p} - \frac{1}{q}$ and $1 \leqslant p < q < \infty$. (For $q = \infty$ a variation of the above was shown.) Extensive use was made of the Nikol'skii inequality for trigonometric polynomials of degree n, t_n

$$\|t_n\|_q \leqslant C n^{\frac{1}{p}-\frac{1}{q}} \|t_n\|_p \quad \text{where} \quad 0 < p \leqslant q \leqslant \infty. \tag{1.3}$$

In [De-Lo, p. 181, Theorem 3.4] (1.1) is proved in a different way with $\omega^r(f,t)_p$ replacing $\omega(f,t)_p$ (which is an improvement) and with 1 replacing q on the right-hand side of (1.1) (which is weaker), and the result is attributed to [De-Ri-Sh] who authored it.

Ul'yanov's result was also extended to the torus $\boldsymbol{T}^d$. There was some effort to extend the result to $\omega_\varphi(f,t)_p$ (see [Ky]), but it involved rearrangements, and while this does work for the extension of (1.2), the modulus of smoothness of a rearrangement may be much smaller than that of the function, and hence leads to a weaker result. To our knowledge, the result for $0 < p < 1$ was not proved in any of the cases. In Section 2, we present the Ul'yanov-type result for $L_p(\boldsymbol{T}^d)$, $0 < p \leqslant q \leqslant \infty$ and in Section 3 for $L_p[-1,1]$ in relation to $\omega_\varphi^r(f,t)_p$, $0 < p < q \leqslant \infty$. We remark on different aspects of the theorems and give some examples of their use. This should be the incentive for the investigation of general results given in Section 4. In Section 5 those general results will be applied to prove the theorems of Sections 2 and 3. In Section 6 we will make some comments on the Nikol'skii-type inequality. In Section 7 the analogous results on $L_p(\boldsymbol{R})$ will be described and proved. In Section 8 the results for best polynomial approximation on simple polytopes are given for $L_p(S)$. The Ul'yanov-type inequality related to approximation with Freud weights will be given in Section 9. Results on K-functionals that measure smoothness on the sphere will be given in Section 10. Results on weighted approximation with Jacobi weights will be given in Section 11. Finally, we mention the paper of Timan [Ti,M], whose nice proof influenced the proof of the crucial Lemma 4.2 in this paper.

2. Ul'yanov-type inequality for $L_p(\boldsymbol{T}^d)$, $0 < p < q \leqslant \infty$

The result of this section is summarized in the following two theorems and will be the model for other results in the paper.

Theorem 2.1. *For $f \in L_p(\boldsymbol{T}^d)$, $0 < p < q \leqslant \infty$ we have for any integer $r \geqslant 1$*

$$\omega^r(f,t)_q \leqslant C \left\{ \int_0^t \left(u^{-\theta} \omega^r(f,u)_p \right)^{q_1} \frac{du}{u} \right\}^{1/q_1} \tag{2.1}$$

and

$$\|f\|_{L_q(T^d)} \leqslant C \left[\left\{ \int_0^1 \left(u^{-\theta} \omega^r(f,u)_p \right)^{q_1} \frac{du}{u} \right\}^{1/q_1} + \|f\|_{L_p(\boldsymbol{T}^d)} \right], \tag{2.2}$$

where $q_1 = \begin{cases} q, & q < \infty \\ 1, & q = \infty \end{cases}, \quad \theta = d\left(\frac{1}{p} - \frac{1}{q}\right)$

and

$$\omega^r(f,u)_p = \sup\left\{\|\Delta_h^r f\|_{L_p(\boldsymbol{T}^d)};\ |h| = \left(h_1^2 + \cdots + h_d^2\right)^{1/2} \leqslant u\right\},$$

$$\Delta_h^r f(x) = \Delta_h\left(\Delta_h^{r-1} f(x)\right) \quad \textit{and} \quad \Delta_h f(x) = f(x+h) - f(x). \tag{2.3}$$

The meaning of (2.1) and (2.2) is that when either of the integrals on the right of (2.1) and of (2.2) (which are well-defined as $f \in L_p(\boldsymbol{T}^d)$) converges, our theorem implies that $f \in L_q(\boldsymbol{T}^d)$, and the inequality in question ((2.1) or (2.2)) is valid. This will be a theme throughout the paper (and will not be commented on again).

In case the reader is puzzled by the jump from $q < \infty$ to $q = \infty$, we observe that this is a common occurrence except when only the weaker result using $q_1 = 1$ whenever $q \geqslant 1$ is proved.

Remark 2.2. The benefit of considering $\omega^r(f,t)_p$ rather than only $\omega(f,t)_p$ (that is, with $r = 1$) becomes evident as

$$\omega^r(f,u)_p = o(u^\theta), \quad u \to 0+ \tag{2.4}$$

is a necessary condition for the integrals on the right of (2.1) and (2.2) to converge, and for $r + \max\left(\frac{1}{p} - 1, 0\right) \leqslant d\left(\frac{1}{p} - \frac{1}{q}\right) = \theta$ (2.4) will imply $\omega^r(f,u)_p = 0$ (in other words $f =$ constant). Summarizing the above, if $r + \max\left(\frac{1}{p} - 1, 0\right) \leqslant d\left(\frac{1}{p} - \frac{1}{q}\right)$, the inequalities (2.1) and (2.2) are trivial, as either the right-hand side diverges ($= \infty$), and is therefore bigger than the left-hand side, or both sides equal zero.

We also prove the following result.

Theorem 2.3. *For* $f \in L_p(\boldsymbol{T}^d)$, $0 < p < q \leqslant \infty$ *we have*

$$E_n(f)_q \leqslant C\left\{\sum_{k=n}^{\infty} k^{q_1\theta - 1} E_k(f)_p^{q_1}\right\}^{1/q_1} \tag{2.5}$$

and

$$\|f\|_q \leqslant C\left[\left\{\sum_{k=1}^{\infty} k^{q_1\theta - 1} E_k(f)_p^{q_1}\right\}^{1/q_1} + \|f\|_p\right], \tag{2.6}$$

where $q_1 = \begin{cases} q, & q < \infty \\ 1, & q = \infty \end{cases}, \quad \theta = d\left(\frac{1}{p} - \frac{1}{q}\right),$

$$E_k(f)_p = \min\left(\|f - T_k\|_p;\ T_k \in \mathcal{T}_k\right), \tag{2.7}$$

and

$$\mathcal{T}_k = \operatorname{span}\left\{e^{i\boldsymbol{k}\cdot\boldsymbol{x}};\ |k_i| \leqslant k,\ \|\boldsymbol{k}\|_{\ell_\infty} \leqslant k\right\}. \tag{2.8}$$

We note that $\mathcal{T}_k$ can be replaced by

$$\mathcal{T}_k^{(\rho)} = \operatorname{span}\left\{e^{i\boldsymbol{k}\cdot\boldsymbol{x}};\ \|\boldsymbol{k}\|_{\ell_\rho} \leqslant k,\ \rho \geqslant 1\right\}$$

and the difference will be only in the constant C in (2.5) and (2.6).

As in Theorem 2.1, the meaning of (2.5) and (2.6) is that if the sum on the right of either will converge, then $f \in L_q(\boldsymbol{T}^d)$, and the inequality in question is valid. This understanding will apply to sums in subsequent sections as well.

We further note that as Theorems 2.1 and 2.3 serve as a model for our further investigations, it is important that we emphasize Theorem 2.3, as in several cases, analogues of Theorem 2.1 are not available but analogues of Theorem 2.3 are. This happens when a proper alternative for $\omega^r(f,t)_p$ eludes us, or when the Jackson-type inequality and the realization result are not known for some p.

3. Ul'yanov-type result using $\omega_\varphi^r(f,t)_p$ and $L_p[-1,1]$

For $f \in L_p[-1,1]$ best polynomial approximation in L_p and $\omega_\varphi^r(f,t)_p$, the Ul'yanov-type inequality is given in the following theorem.

Theorem 3.1. *For $f \in L_p[-1,1]$, $0 < p < q \leqslant \infty$ we have for any integer $r \geqslant 1$*

$$\omega_\varphi^r(f,t)_q \leqslant C\left(\int_0^t \left(u^{-\theta}\omega_\varphi^r(f,u)_p\right)^{q_1}\frac{du}{u}\right)^{1/q_1}, \tag{3.1}$$

$$\|f\|_{L_q[-1,1]} \leqslant C\left[\left\{\int_0^1 \left(u^{-\theta}\omega_\varphi^r(f,u)_p\right)^{q_1}\frac{du}{u}\right\}^{1/q_1} + \|f\|_{L_p[-1,1]}\right], \tag{3.2}$$

$$E_n(f)_q \leqslant C\left\{\sum_{k=n}^{\infty} k^{q_1\theta-1}E_k(f)_p^{q_1}\right\}^{1/q_1}, \tag{3.3}$$

and

$$\|f\|_{L_q[-1,1]} \leqslant C\left[\left\{\sum_{k=1}^{\infty} k^{q_1\theta-1}E_k(f)_p^{q_1}\right\}^{1/q_1} + \|f\|_{L_p[-1,1]}\right], \tag{3.4}$$

where $q_1 = \begin{cases} q, & q<\infty \\ 1, & q=\infty \end{cases}$, $\quad \theta = 2\left(\frac{1}{p}-\frac{1}{q}\right)$,

$$\omega_\varphi^r(f,t)_p = \sup_{|h|\leqslant t}\|\Delta_{h\varphi}^r f\|_{L_p[-1,1]} \tag{3.5}$$

with

$$\Delta_{h\varphi}^r f(x) = \begin{cases} \sum_{k=0}^{r}(-1)^k\binom{r}{k} f\left(x+\left(\frac{r}{2}-k\right)h\varphi\right), & x \pm \frac{r}{2}h\varphi \in [-1,1] \\ 0 & \textit{otherwise} \end{cases},$$

$$\varphi = \varphi(x) = \sqrt{1-x^2}$$

and

$$E_n(f)_p = \inf\left(\|f-P\|_{L_p[-1,1]};\ P \text{ is a polynomial of degree } n\right). \tag{3.6}$$

Remark 3.2. To give examples, we note that if $f(x) = (1-x^2)^{-1/2}$, simple calculations (see also [Di-To, pp. 34–35]) show that $\omega_\varphi(f,t)_1 = O(t|\log t|)$ and $\omega_\varphi^2(f,t)_1 = O(t)$, and either estimate implies via (3.2) that $f \in L_q$ for $q < 2$ but does not imply that $f \in L_2$. In fact, $f \notin L_2$ and this shows that $\theta = 2\left(\frac{1}{p}-\frac{1}{q}\right)$ cannot be improved. Further, if $f(x) = (1-x^2)^{-1/3}$, $\omega_\varphi(f,t)_1 = O(t)$ and $\omega_\varphi^2(f,t)_1 = O(t^{4/3})$. Using (3.2) with $r = 1$, we get $f \in L_q$ only for $q < 2$, but using (3.2) with $r = 2$, we have $f \in L_q$ for $q < 3$, which shows the advantage of using $r > 1$ in our theorem. It is clear that $f \notin L_3[-1,1]$. Finally, for $f(x) = (1-x^2)^{-1/2}|\log(1-x^2)|^\gamma$, we have $\omega_\varphi^2(f,t)_1 = O(t\,|\log t|^\gamma)$. We set $\gamma = -1$ and (3.2) with $q_1 = 2$ implies $f \in L_2[-1,1]$, as is in fact the case. However, if we used (3.2) with $q_1 = 1$ (instead of $q_1 = 2$), we could not have deduced $f \in L_2[-1,1]$, which shows the benefit of using the power $q_1 = q$ (and not 1), in estimates (3.1)–(3.4).

We note here that in the above examples we could have used (3.1), (3.3) or (3.4) to show the benefits of the different parameters, but we chose (3.2) for simplicity.

4. Ul'yanov-type result, general framework

Let $L_{p,w}(\mathcal{D})$ be the collection of functions on $\mathcal{D}$ satisfying

$$\|f\|_p = \|f\|_{L_{p,w}(\mathcal{D})} \equiv \left\{\int_{\mathcal{D}} |f|^p w\, dx\right\}^{1/p} < \infty \tag{4.1}$$

for the given p, $0 < p < \infty$ where $\mathcal{D}$ is a measurable set and $w(x) > 0$ except perhaps on the boundary of $\mathcal{D}$ which is of measure 0. We also set as usual $f \in L_{\infty,w}(\mathcal{D}) = L_\infty(\mathcal{D})$.

We note that in the applications of the above given in this paper $\mathcal{D}$ will be $\boldsymbol{T}$, $\boldsymbol{T}^d$, $\boldsymbol{R}$, $[-1,1]$, a simple polytope or the sphere; and the weight w will most times be $w(x) = 1$, but we will also use Freud's weight on $\boldsymbol{R}$, the Jacobi weight on $[-1,1]$ or on the simplex.

In the following $\{\mathcal{A}_\sigma\}_{\sigma\in\mathcal{O}}$ is a collection of linear subspaces of $L_{p,w}(\mathcal{D})$ with $\mathcal{O} \subset \boldsymbol{R}_+$ satisfying

$$\mathcal{A}_\sigma \subset L_{p,w}(\mathcal{D}) \quad \text{for all } \sigma \in \mathcal{O}, \quad \mathcal{A}_\sigma \subset \mathcal{A}_{\sigma_1} \quad \text{for} \quad \sigma < \sigma_1 \tag{4.2}$$

and $\bigcup_{\sigma\in\mathcal{O}} \mathcal{A}_\sigma$ is dense in $L_{p,w}(\mathcal{D})$, $0 < p < \infty$.

In applications we will write $\mathcal{A}_n$ when $\mathcal{O}$ is the set of positive integers, for example when discussing trigonometric polynomials on $\boldsymbol{T}$ or $\boldsymbol{T}^d$, algebraic polynomials of total degree n and spherical polynomials of degree n. We can also have $\mathcal{A}_\sigma$ with $\sigma \in \mathcal{O}$, which has a continuous parameter σ like exponential functions of type σ on $\boldsymbol{R}$. It can be noted that in the applications below when we specify that $\mathcal{O} \subset \boldsymbol{N}$ and, we write $\mathcal{A}_n$, $\mathcal{A}_n$ will be a finite-dimensional space (not necessarily n-dimensional), but $\mathcal{A}_\sigma$ when $\mathcal{O} = [a, \infty)$ will not necessarily be finite-dimensional.

Definition 4.1. The collection $\{\mathcal{A}_\sigma\}_{\sigma\in\mathcal{O}}$ belongs to the Nikol'skii class $\mathcal{N}(\beta)$ if

$$\|\varphi\|_{L_{q,w}(\mathcal{D})} \leqslant C\sigma^{\frac{\beta}{p}-\frac{\beta}{q}} \|\varphi\|_{L_{p,w}(\mathcal{D})} \quad \text{for } \varphi \in \mathcal{A}_\sigma, \quad 0 < p \leqslant q \leqslant \infty \quad \text{and all} \quad \sigma \in \mathcal{O}. \tag{4.3}$$

It is understood that for $\{\mathcal{A}_\sigma\}$ to belong to $\mathcal{N}(\beta)$ C in (4.3) is independent of $\sigma \in \mathcal{O}$ but may depend on p and q.

Definition 4.2. The rate of best approximation is

$$E_\sigma(f)_p = \inf_{\varphi\in\mathcal{A}_\sigma} \|f - \varphi\|_{L_{p,w}(\mathcal{D})}, \tag{4.4}$$

and the best approximant φ_σ from $\mathcal{A}_\sigma$ to f in $L_{p,w}(\mathcal{D})$ is given by

$$\|\varphi_\sigma - f\|_{L_{p,w}(\mathcal{D})} = E_\sigma(f)_p. \tag{4.5}$$

In the following we will assume that the best approximant exists for $0 < p < \infty$. In all applications below the existence and uniqueness of φ_σ are achieved for $0 < p < \infty$. We note that for the purpose of the proof, however, the existence of a near best approximant

$$\|\varphi_\sigma - f\|_{L_{p,w}(\mathcal{D})} \leqslant A E_\sigma(f)_p, \tag{4.5$'$}$$

where A does not depend on σ, is sufficient.

We are now able to state and prove the general analogue of Theorem 2.3 and inequalities (3.3) and (3.4).

Theorem 4.1. *For $f \in L_{p,w}(\mathcal{D})$, $0 < p < q \leqslant \infty$, a collection of linear spaces $\mathcal{A}_\sigma$ that belong to $\mathcal{N}(\beta)$ (that is, satisfying (4.3)), we have in case $\mathcal{A}_\sigma$ is given for all $\sigma \in [1, \infty)$ ($\mathcal{O} = [1, \infty)$)*

$$E_\sigma(f)_q \leqslant C \left\{ \int_\sigma^\infty v^{q_1\theta-1} E_v(f)_p^{q_1}\, dv \right\}^{1/q_1} \tag{4.6}$$

and

$$\|f\|_q \leqslant C \left[\left\{ \int_1^\infty v^{q_1\theta-1} E_v(f)_p^{q_1}\, dv \right\}^{1/q_1} + \|f\|_p \right], \tag{4.7}$$

where $q_1 = \begin{cases} q, & 0 < q < \infty \\ 1, & q = \infty \end{cases}$ *and* $\theta = \beta\left(\frac{1}{p} - \frac{1}{q}\right)$. *Similarly, if $\mathcal{O} = \mathbf{N}$ and $\mathcal{A}_n$ belongs to $\mathcal{N}(\beta)$, we have*

$$E_n(f)_q \leqslant C \left\{ \sum_{k=n}^\infty k^{q_1\theta-1} E_k(f)_p^{q_1} \right\}^{1/q_1} \tag{4.6$'$}$$

and

$$\|f\|_q \leqslant C\left[\left\{\sum_{k=1}^{\infty} k^{q_1\theta-1} E_k(f)_p^{q_1}\right\}^{1/q_1} + \|f\|_p\right]. \tag{4.7$'$}$$

We need the following crucial lemma:

Lemma 4.2. *Under the conditions of Theorem 4.1 and with φ_σ of (4.5) or (4.5)$'$*

$$\left\|\sum_{\ell=1}^{m}(\varphi_{\sigma 2^\ell}-\varphi_{\sigma 2^{\ell-1}})\right\|_q \leqslant C\left(\sum_{\ell=1}^{m}\left((\sigma 2^\ell)^{\beta(\frac{1}{p}-\frac{1}{q})} E_{\sigma 2^{\ell-1}}(f)_p\right)^{q_1}\right)^{1/q_1} \tag{4.8}$$

with $C \equiv C(p, q, \beta)$ independent of m.

We will use (4.8) with a general $\sigma \in [1, \infty)$ or with $\sigma = n$ or $\sigma = 1$ on different occasions.

Proof. For $q \leqslant 1$ $(q_1 = q)$ we write

$$\begin{aligned}\left\|\sum_{\ell=1}^{m}\left(\varphi_{\sigma 2^\ell}-\varphi_{\sigma 2^{\ell-1}}\right)\right\|_q^q &\leqslant \sum_{\ell=1}^{m}\|\varphi_{\sigma 2^\ell}-\varphi_{\sigma 2^{\ell-1}}\|_q^q \\ &\leqslant C\sum_{\ell=1}^{m}\left(\sigma 2^\ell\right)^{\beta(\frac{1}{p}-\frac{1}{q})q}\|\varphi_{\sigma 2^\ell}-\varphi_{\sigma 2^{\ell-1}}\|_p^q \\ &\leqslant C\sum_{\ell=1}^{m}\left(\sigma 2^\ell\right)^{\beta(\frac{1}{p}-\frac{1}{q})q} E_{\sigma 2^{\ell-1}}(f)_p^q.\end{aligned}$$

For $q \geqslant 1$ and $q_1 = 1$, we write

$$\begin{aligned}\left\|\sum_{\ell=1}^{m}\left(\varphi_{\sigma 2^\ell}-\varphi_{\sigma 2^{\ell-1}}\right)\right\|_q &\leqslant \sum_{\ell=1}^{m}\|\varphi_{\sigma 2^\ell}-\varphi_{\sigma 2^{\ell-1}}\|_q \\ &\leqslant C\sum_{\ell=1}^{m}\left(\sigma 2^\ell\right)^{\beta(\frac{1}{p}-\frac{1}{q})}\|\varphi_{\sigma 2^\ell}-\varphi_{\sigma 2^{\ell-1}}\|_p \\ &\leqslant 2C\sum_{\ell=1}^{m}\left(\sigma 2^\ell\right)^{\beta(\frac{1}{p}-\frac{1}{q})} E_{\sigma 2^{\ell-1}}(f)_p.\end{aligned}$$

In fact, we need $q_1 = 1$ only for $q = \infty$ but the above can provide an easier proof of Theorem 4.1 if $q_1 = 1$ is assumed for $q \geqslant 1$. To complete the proof we need to settle the case $1 < q < \infty$ and $q_1 = q$, which is the hard part. We follow the idea of the proof in [Ti,M]. We set $\eta_\ell = \eta_\ell(x) \equiv |\varphi_{\sigma 2^\ell}(x) - \varphi_{\sigma 2^{\ell-1}}(x)|$, and choosing $r = [q] + 1$ (recall $1 < q < \infty$), we have

$$I(m) \equiv \left\|\sum_{\ell=1}^{m}\left(\varphi_{\sigma 2^\ell}-\varphi_{\sigma 2^{\ell-1}}\right)\right\|_q$$

$$\leqslant \left[\int \left(\sum_{\ell=1}^{m} \eta_\ell \right)^q w \right]^{1/q}$$
$$\leqslant \left[\int \left(\sum_{\ell=1}^{m} \eta_\ell^{q/r} \right)^r w \right]^{1/q}$$
$$= \left[\sum_{\ell_1=1}^{m} \cdots \sum_{\ell_r=1}^{m} \int \eta_{\ell_1}^{q/r} \cdots \eta_{\ell_r}^{q/r} w \right]^{1/q}.$$

We note that

$$\left(\prod_{n=1}^{r} a_n \right)^{r-1} = \prod_{1 \leqslant i < j \leqslant r} a_i a_j \quad \text{for} \quad r > 1$$

which follows from the observation that on the right-hand side every a_n appears exactly $r - 1$ times. Hence we obtain

$$I(m) \leqslant \left[\sum_{\ell_1=1}^{m} \cdots \sum_{\ell_r=1}^{m} \int \left(\prod_{1 \leqslant i < j \leqslant r} \eta_{\ell_i}^{q/r} \eta_{\ell_j}^{q/r} \right)^{1/(r-1)} w \right]^{1/q}.$$

We now use the extended (or generalized) Hölder inequality (see [Zy, (9.8), p. 18] or [He-St, 13.26, p. 200]) given by

$$\int g_1 \cdots g_n \leqslant \|g_1\|_{1/\alpha_1} \cdots \|g_n\|_{1/\alpha_n}, \quad \alpha_k > 0, \quad \sum_{i=1}^{n} \alpha_k = 1.$$

This implies with $\alpha_k = \frac{2}{r(r-1)}$ where $k = 1, \ldots, n$, $n = \frac{r(r-1)}{2}$, k corresponds to the pair (i, j) $i < j$ ordered lexicographically and $g_k = \eta_{\ell_i}^{\frac{q}{r(r-1)}} \eta_{\ell_j}^{\frac{q}{r(r-1)}}$

$$I(m) \leqslant \left[\sum_{\ell_1=1}^{m} \cdots \sum_{\ell_r=1}^{m} \prod_{1 \leqslant i < j \leqslant r} \left(\int \eta_{\ell_i}^{q/2} \eta_{\ell_j}^{q/2} w \right)^{\frac{2}{r(r-1)}} \right]^{1/q}.$$

We define $J(\ell_i, \ell_j)$ by

$$J(\ell_i, \ell_j) \equiv \int \eta_{\ell_i}^{q/2} \eta_{\ell_j}^{q/2} w.$$

To estimate $J(\ell_i, \ell_j)$ we use the Hölder inequality with powers $\alpha = \frac{p+q}{p}$ and $\alpha' = \frac{p+q}{q}$ $\left(\alpha^{-1} + (\alpha')^{-1} = 1\right)$ and write

$$J(\ell_i, \ell_j) \leqslant \left(\int \eta_{\ell_i}^{\frac{(p+q)q}{2p}} w \right)^{p/(p+q)} \left(\int \eta_{\ell_j}^{\frac{p+q}{2}} w \right)^{q/(p+q)}.$$

Observing that $q_2 = \frac{(p+q)q}{2p} > p$ and $q_3 = \frac{p+q}{2} > p$, we recall that $\eta_{\ell_i} \in \mathcal{A}_{\sigma 2^{\ell_i}}$, $\eta_{\ell_j} \in \mathcal{A}_{\sigma 2^{\ell_j}}$, and using Nikol'skii's inequality, we obtain

$$\begin{aligned} J(\ell_i, \ell_j) &\leqslant \|\eta_{\ell_i}\|_{q_2}^{q/2} \cdot \left\|\eta_{\ell_j}\right\|_{q_3}^{q/2} \\ &\leqslant C\left[\left(\sigma 2^{\ell_i}\right)^{\beta(\frac{1}{p}-\frac{1}{q_2})} \|\eta_{\ell_i}\|_p (\sigma 2^{\ell_j})^{\beta(\frac{1}{p}-\frac{1}{q_3})} \|\eta_{\ell_j}\|_p\right]^{q/2} \\ &= C\left[\left(\sigma 2^{\ell_i}\right)^{\beta(\frac{1}{p}-\frac{1}{q})} \|\eta_{\ell_i}\|_p \left(\sigma 2^{\ell_j}\right)^{\beta(\frac{1}{p}-\frac{1}{q})} \|\eta_{\ell_j}\|_p\right]^{q/2} \cdot \left(2^{(\ell_i-\ell_j)}\right)^{\frac{(q-p)\beta}{2(p+q)}}. \end{aligned}$$

Symmetry between i and j in $J(\ell_i, \ell_j)$ allows us to exchange i and j if $\ell_i > \ell_j$ and replace $(2^{(\ell_i-\ell_j)})^{(q-p)\beta/2(p+q)}$ by $(2^{-|\ell_j-\ell_i|})^{(q-p)\beta/2(p+q)}$. Hence we have

$$\begin{aligned} I(m) \leqslant C_1 \Bigg[\sum_{\ell_1=1}^{m} \cdots \sum_{\ell_r=1}^{m} \Bigg\{ \prod_{1\leqslant i<j\leqslant r} &\left(\left(\sigma 2^{\ell_i}\right)^{\beta(\frac{1}{p}-\frac{1}{q})} \|\eta_{\ell_i}\|_p \right. \\ &\left. \times (\sigma 2^{\ell_j})^{\beta(\frac{1}{p}-\frac{1}{q})} \|\eta_{\ell_j}\|_p\right)^q 2^{-|\ell_i-\ell_j| \frac{(q-p)\beta}{p+q}} \Bigg\}^{\frac{1}{r(r-1)}} \Bigg]^{1/q}. \end{aligned}$$

We use the identity

$$\prod_{1\leqslant i<j\leqslant r} a_{\ell_i} a_{\ell_j} 2^{-|\ell_i-\ell_j|\gamma} = \prod_{s=1}^{r} a_{\ell_s}^{r-1} \prod_{k=1}^{r} 2^{-|\ell_s-\ell_k|\gamma/2}$$

for $a_{\ell_s} \geqslant 0$, $1 \leqslant s \leqslant r$ and $\gamma > 0$.

Setting

$$a_{\ell_s} = \left((\sigma 2^{\ell_s})^{\beta(\frac{1}{p}-\frac{1}{q})} \|\eta_{\ell_s}\|_p\right)^{\frac{q}{r(r-1)}}$$

and $\gamma = \frac{(q-p)\beta}{(p+q)r(r-1)}$, we obtain

$$\begin{aligned} I(m) \leqslant C_1 \Bigg[\sum_{\ell_1=1}^{m} \cdots \sum_{\ell_r=1}^{m} \prod_{s=1}^{r} &\left((\sigma 2^{\ell_s})^{\beta(\frac{1}{p}-\frac{1}{q})} \|\eta_{\ell_s}\|_p\right)^{q/r} \\ &\times \prod_{k=1}^{r} 2^{-|\ell_s-\ell_k|(q-p)\beta/(2(p+q)r(r-1))} \Bigg]^{1/q}. \end{aligned}$$

We recall the extended (or generalized) Hölder inequality for sums (see [Zy, (9.8), p. 18] or [He-St, 13.26, p. 200]) given by

$$\sum_{v} a_v(1) \cdots a_v(r) \leqslant \left(\sum_{v} |a_v(1)|^{\frac{1}{\alpha_1}}\right)^{\alpha_1} \cdots \left(\sum_{v} |a_v(r)|^{\frac{1}{\alpha_r}}\right)^{\alpha_r},$$

where $\alpha_k > 0$ and $\sum_{k=1}^{r} \alpha_k = 1$.

We now use this with the sum $\sum_{\ell_1=1}^{m} \cdots \sum_{\ell_1=1}^{m}$ (with v is $(\ell_1, \dots, \ell_r)$ ordered lexicographically) and with $\alpha_k = \frac{1}{r}$ to obtain

$$I(m) \leqslant C_1 \left(\prod_{s=1}^{r} \left[\sum_{\ell_1=1}^{m} \cdots \sum_{\ell_r=1}^{m} \left(\|\eta_{\ell_s}\|_p (\sigma 2^{\ell_s})^{\beta(\frac{1}{p}-\frac{1}{q})} \right)^q \right.\right.$$
$$\left.\left. \times \prod_{k=1}^{r} 2^{-|\ell_s-\ell_k|\beta(q-p)/(2(p+q)(r-1))} \right]^{1/r} \right)^{1/q}.$$

We now observe that all r factors of the product of the last expression are equal and the common value is

$$A(m) \equiv \left[\sum_{\ell_1=1}^{m} \left((\sigma 2^{\ell_1})^{\beta(\frac{1}{p}-\frac{1}{q})} \|\eta_{\ell_1}\|_p \right)^q \right.$$
$$\left. \times \sum_{\ell_2=1}^{m} \cdots \sum_{\ell_r=1}^{m} \prod_{k=1}^{r} 2^{-|\ell_1-\ell_k|\beta(q-p)/(2(p+q)(r-1))} \right]^{1/r}.$$

By the inequality

$$\sum_{\ell=1}^{m} 2^{-|\ell-\ell_1|\gamma} \leqslant 2 \sum_{\ell=0}^{\infty} 2^{-\ell\gamma} \equiv C(\gamma) \quad \forall \ell_1 \in \mathbf{N}, \quad \gamma > 0$$

we have

$$\sum_{\ell_2=1}^{m} \cdots \sum_{\ell_r=1}^{m} \prod_{k=1}^{r} 2^{-|\ell_k-\ell_1|\gamma} = \sum_{\ell_2=1}^{m} \cdots \sum_{\ell_r=1}^{m} \prod_{k=2}^{r} 2^{-|\ell_k-\ell_1|\gamma}$$
$$\leqslant \prod_{k=2}^{r} \left(\sum_{\ell_k=1}^{m} 2^{-|\ell_k-\ell_1|\gamma} \right) \leqslant (C(\gamma))^{r-1}.$$

Therefore, we have

$$I(m) \leqslant C_1 (A(m))^{r/q}$$
$$\leqslant C_2 \left(\sum_{\ell=1}^{m} (\sigma 2^{\ell})^{\beta(\frac{1}{p}-\frac{1}{q})q} \|\eta_\ell\|_p^q \right)^{1/q},$$

where $C_2 = C_1 \left\{ C \left(\frac{\beta(q-p)}{2(p+q)(r-1)} \right) \right\}^{\frac{r-1}{q}}$ which does not depend on m.

Recalling $\|\eta_\ell\|_p \leqslant 2E_{\sigma 2^{\ell-1}}(f)_p$, we complete the proof. □

Proof of Theorem 4.1. As $\bigcup \mathcal{A}_\sigma$ (or $\bigcup \mathcal{A}_n$) is dense in $L_{p,w}(\mathcal{D})$, we choose φ_σ or φ_n by (4.5) and $\varphi_{\sigma 2^m} - \varphi_\sigma$ or $\varphi_{n2^m} - \varphi_n$ tends in L_p, and therefore in measure locally, to

$f-\varphi_\sigma$ of $f-\varphi_n$ respectively. If the best approximant does not exist, we choose a near best approximant as described in (4.5)′ before the statement of Theorem 4.1 with a fixed constant A ($A=2$ for example). While such a situation does not occur in the applications given in this paper, we did not want to burden any theorem with an extra condition. Hence, if convergence can be shown in L_q, $\varphi_{\sigma 2^m}-\varphi_\sigma$ or $\varphi_{n2^m}-\varphi_n$ tends in $L_{q,w}(\mathcal{D})$ to $f-\varphi_\sigma$ or $f-\varphi_n$ respectively for $f\in L_{q,w}(D)$. Using Lemma 4.2, we have

$$\begin{aligned}\|f-\varphi_\sigma\|_q &\leqslant \lim_{m\to\infty}\|\varphi_{\sigma 2^m}-\varphi_\sigma\|_q\\ &= \lim_{m\to\infty}\left\|\sum_{\ell=1}^{m}\left(\varphi_{2^\ell\sigma}-\varphi_{2^{\ell-1}\sigma}\right)\right\|_q\\ &\leqslant C\lim_{m\to\infty}\left(\sum_{\ell=1}^{m}\left(\left(\sigma 2^\ell\right)^{\beta(\frac{1}{p}-\frac{1}{q})}E_{\sigma 2^{\ell-1}}(f)_p\right)^{q_1}\right)^{1/q_1}\\ &\leqslant C\left(\sum_{\ell=1}^{\infty}\left(\left(\sigma 2^\ell\right)^{\beta(\frac{1}{p}-\frac{1}{q})}E_{\sigma 2^{\ell-1}}(f)_p\right)^{q_1}\right)^{1/q_1}.\end{aligned}$$

Monotonicity of $E_\sigma(f)_p$ or $E_n(f)_p$ implies that the last sum is bounded by the right-hand side of (4.6) or (4.6)′ for any σ or n. We note that when proving (4.7) or (4.7)′, we use $\sigma=1$ or $n=1$.

To prove (4.7) we write $\|f\|_q\leqslant\|f-\varphi_1\|_q+\|\varphi_1\|_q$ for $q\geqslant 1$ and $\|f\|_q^q\leqslant\left(\|f-\varphi_1\|_q^q+\|\varphi_1\|_q^q\right)$ for $0<q<1$, and complete the proof observing that $\|\varphi_1\|_q\leqslant C\|\varphi_1\|_p$ with C of the Nikol'skii inequality and $\|\varphi_1\|_p\leqslant\|f-\varphi_1\|_p+\|f\|_p\leqslant E_1(f)_p+\|f\|_p$ for $p\geqslant 1$ while $\|\varphi_1\|_p^p\leqslant\|f-\varphi_1\|_p^p+\|f\|_p^p\leqslant E_1(f)_p^p+\|f\|_p^p$ for $0<p<1$.

Finally, (4.6) or (4.6)′ follows from the above estimates and $E_\sigma(f)_q\leqslant A\|f-\varphi_\sigma\|_q$ or $E_n(f)_q\leqslant A\|f-\varphi_n\|_q$. □

For the general form of the Ul'yanov-type result one needs also the following two theorems. We will use these theorems in the proof of Theorems 2.1 and 3.1 as well as for many results in subsequent sections.

In the following two theorems various Jackson and Bernstein-type inequalities as well as realization results will be used. These, together with the Nikolskii-type inequality used in Theorem 4.1, are crucial for the setup needed for proving the full analogue of the Ul'yanov type inequality.

Theorem 4.3. *Suppose in addition to the assumptions in Theorem 4.1 there exists an increasing function on* $[0,1]$ $\Omega(f,t)$ *satisfying*

$$E_\sigma(f)_p\leqslant C\Omega\left(f,\frac{1}{\sigma}\right)_p \quad \textit{for all} \quad \sigma\in[1,\infty) \quad \textit{or all} \quad \sigma\in\mathbf{N}. \tag{4.9}$$

Then for $0 < p < q \leqslant \infty$

$$\|f\|_q \leqslant C_1 \left[\left\{ \int_0^1 \left(u^{-\theta} \Omega(f,t)_p \right)^{q_1} \frac{du}{u} \right\}^{1/q_1} + \|f\|_p \right] \tag{4.10}$$

or

$$\|f\|_q \leqslant C_1 \left[\left\{ \sum_{k=1}^{\infty} k^{q_1\theta-1} \Omega\left(f, \frac{1}{k}\right)^{q_1} \right\}^{1/q_1} + \|f\|_p \right] \tag{4.10$'$}$$

with $q_1 = \begin{cases} q & \text{for} \quad q < \infty \\ 1 & \text{for} \quad q = \infty \end{cases}$ *and* $\theta = \beta\left(\frac{1}{p} - \frac{1}{q}\right)$.

Proof. We substitute (4.9) in (4.7) and change variable $u = \frac{1}{v}$. Or just substitute (4.9) with $\sigma = k$ in (4.7)$'$ to obtain (4.10)$'$. □

We observe that the Jackson-type inequality (4.9) is assumed only for one p in this theorem and relates only to that p. In applications usually if a Jackson-type inequality is proved for p, similar results follow for p_1 satisfy $p \leqslant p_1 \leqslant \infty$.

Minor modifications to (4.10) and (4.10)$'$ will be necessary if in (4.9) $\sigma \in [a, \infty)$ or $\sigma \in \boldsymbol{N}$ with $\sigma \geqslant r$ is assumed respectively.

The final part of the Ul'yanov-type result was separated because some additional conditions were still needed, and we attempted to separate the conditions so that it is clear which conditions are needed for which part of the result. In many applications all these conditions are satisfied.

Theorem 4.4. *Suppose in addition to the assumptions of Theorems 4.1 and 4.3 we have an increasing function* $\Omega(f,t)_q \equiv \Omega^\gamma(f,t)_q$, $t \in (0,\infty)$ *satisfying*

$$\Omega^\gamma\left(f, \frac{1}{\sigma}\right)_q \leqslant C \left(\|f - \varphi_\sigma\|_q + \sigma^{-\gamma} \Phi(\varphi_\sigma)_q \right) \tag{4.11}$$

for $\sigma \in \mathcal{O}$, $\varphi_\sigma \in \mathcal{A}_\sigma$ *and a seminorm (for* $q \geqslant 1$) *or quasi seminorm (for* $0 < q < 1$) $\Phi(\varphi_\sigma)_q$. *We suppose further that*

$$\Phi(\varphi_\sigma)_q \leqslant C\sigma^{\beta(\frac{1}{p}-\frac{1}{q})} \Phi(\varphi_\sigma)_p \quad \text{for} \quad \sigma \in \mathcal{O}, \ \varphi_\sigma \in \mathcal{A}_\sigma \tag{4.12}$$

and that for φ_σ *satisfying* (4.5)

$$\sigma^{-\gamma} \Phi(\varphi_\sigma)_p \leqslant C\Omega^\gamma\left(f, \frac{1}{\sigma}\right)_p \equiv C\Omega\left(f, \frac{1}{\sigma}\right)_p, \tag{4.13}$$

where C *in both* (4.12) *and* (4.13) *is independent of* ξ. *Then for* $t \leqslant 1$ (*or* $t = \frac{1}{n}$ *when* $\mathcal{O} \equiv \boldsymbol{N}$)

$$\Omega^\gamma(f,t)_q \leqslant C \left\{ \int_0^{2t} \left(u^{-\theta} \Omega^\gamma(f,u)_p \right)^{q_1} \frac{du}{u} \right\}^{1/q_1} \tag{4.14}$$

where

$$q_1 = \begin{cases} q & \text{for} \quad 0 < q < \infty \\ 1 & \text{for} \quad q = \infty \end{cases} \qquad \text{and} \qquad \theta = \beta\left(\frac{1}{p} - \frac{1}{q}\right).$$

In applications we will have

$$\sup_{\xi \in \Xi} \|P_\xi(D)\varphi_\sigma\|_p \equiv \Phi(\varphi_\sigma)_p, \tag{4.15}$$

where Ξ is a set which most times will be a singleton (see Sections 5, 7, 9–11), sometimes a finite set (see Section 8) and sometimes even an uncountable set (like $\xi \in \{\xi \in \boldsymbol{R}^d : |\xi| = 1\}$ and $P_\xi(D) = (\frac{\partial}{\partial \xi})^r$ in Section 5). The linear operator $P_\xi(D)$ will represent in most cases a differential operator. It may represent an operator related to a differential operator like $\widetilde{f}'$ or a fractional power of a differential operator for example.

We observe that the Jackson-type result is still assumed only for p (see (4.9)) and that the same is true for the more difficult part of the realization result, that is, (4.13). What is usually the easy direction of a realization result, that is (4.11), is assumed only for q. However, in the applications given in this paper results like (4.9), (4.11)–(4.13) and others are valid for a wide range of p and q with more properties than required. We could have replaced the range of the integration in (4.14) by $[0, t]$ if we made further easy assumptions on $\Omega^\gamma(f, t)_p$.

Proof. To estimate $\Omega^\gamma(f, t)_q$ using (4.11) any $\varphi_\sigma \in \mathcal{A}_\sigma$ will do, and we choose φ_σ satisfying (4.5). Following the proof of Theorem 4.1 with the φ_σ that satisfies (4.5), we get

$$\|f - \varphi_\sigma\|_q \leqslant C \left\{ \int_0^{1/\sigma} \left(u^{-\theta}\Omega^\gamma(f, u)_p\right)^{q_1} \frac{du}{u} \right\}^{1/q_1}.$$

To estimate the second term we combine (4.12) with (4.13) and obtain

$$\begin{aligned}
\sigma^{-\gamma}\Phi(\varphi_\sigma)_q &\leqslant C\sigma^{-\gamma}\sigma^{\beta(\frac{1}{p}-\frac{1}{q})}\,\Phi(\varphi_\sigma)_p \\
&\leqslant C\sigma^{\beta(\frac{1}{p}-\frac{1}{q})}\Omega^\gamma\left(f, \frac{1}{\sigma}\right)_p \\
&\leqslant C\left\{\int_{1/\sigma}^{2/\sigma} \left(u^{-\theta}\Omega^\gamma(f, u)_p\right)^{q_1} \frac{du}{u}\right\}^{1/q_1},
\end{aligned}$$

which completes the proof. □

5. Trigonometric and algebraic polynomials in $L_p(\boldsymbol{T}^d)$ and $L_p[-1, 1]$ respectively

In this section we prove Theorems 2.1, 2.3 and 3.1, which are the model and motivation for other results of this paper.

Proof of Theorem 2.3. We use Theorem 4.1 in which we set $L_{p,w}(\mathcal{D}) = L_p(\boldsymbol{T}^d)$, $\mathcal{O} = \boldsymbol{N}$, $\mathcal{A}_n = \mathcal{T}_n$ with $\mathcal{T}_n$ of (2.8) and $\beta = d$. We note that assumption (4.3) that is made in

Theorem 4.1, that is $\mathcal{A}_n \in \mathcal{N}(\beta)$ now takes the form

$$\|T_n\|_{L_q(\boldsymbol{T}^d)} \leqslant C n^{d(\frac{1}{p}-\frac{1}{q})} \|T_n\|_{L_p(\boldsymbol{T}^d)}, \quad 0 < p \leqslant q \leqslant \infty, \quad T_n \in \mathcal{T}_n \tag{5.1}$$

which is the classical Nikol'skii inequality (see for $1 \leqslant p \leqslant \infty$ [Ni] and for $0 < p < 1$ [De-Lo, p. 102]). The density of trigonometric polynomials in $L_p(\boldsymbol{T}^d)$, $0 < p < \infty$ is also well-established. Therefore, all the assumptions of Theorem 4.1 are fulfilled in this setup and we obtain Theorem 2.3. $\square$

Proof of Theorem 2.1. We will use Theorems 4.3 and 4.4 to prove (2.2) and (2.1) respectively. We set in both theorems $\Omega^\gamma(f,t)_p = \omega^r(f,t)_p$, (for q as well) with $\omega^r(f,t)_p$ given by (2.3). We recall the classical Jackson-type estimate

$$E_n(f)_p \leqslant C\omega^r\left(f, \frac{1}{n}\right)_p, \quad 0 < p \leqslant \infty \tag{5.2}$$

with $E_n(f)_p$ of (2.7). Inequality (5.2) is (4.9) of Theorem 4.3 in our case. Therefore, the conditions in Theorem 4.3 are satisfied and (2.2) follows from (4.10).

To prove (2.1) we note that for any $T_n \in \mathcal{T}_n$ and $0 < q \leqslant \infty$

$$\begin{aligned}\omega^r(f,t)_q &\leqslant \omega^r(f-T_n,t)_q + \omega^r(T_n,t)_q \\ &\leqslant C\left[\|f-T_n\|_q + t^r \sup_\xi \left\|\left(\frac{\partial}{\partial\xi}\right)^r T_n\right\|_q\right],\end{aligned}$$

which is (4.11) with $\gamma = r$ and $\Phi(T_n)_q = \sup_\xi \|P_\xi(D)T_n\|_q = \sup_\xi \|\left(\frac{\partial}{\partial\xi}\right)^r T_n\|_q$ (see also (4.15)). As $\left(\frac{\partial}{\partial\xi}\right)^r T_n \in \mathcal{T}_n$ if $T_n \in \mathcal{T}_n$, (4.12) is satisfied with $\sigma = n$, $\gamma = r$, $\beta = d$ and $P_\xi(D) = \left(\frac{\partial}{\partial\xi}\right)^r$. Inequality (4.13) follows from the equivalence given by the realization result

$$\omega^r\left(f, \frac{1}{n}\right)_p \approx \|f-T_n\|_p + n^{-r} \sup_\xi \left\|\left(\frac{\partial}{\partial\xi}\right)^r T_n\right\|_p, \tag{5.3}$$

which is valid for $0 < p \leqslant \infty$ and T_n satisfying $\|f - T_n\|_p = E_n(f)_p$. We note that sometimes the realization result is written as (see for the one-dimensional case [Di-Hr-Iv, Theorem 3.1])

$$\omega^r\left(f, \frac{1}{n}\right)_p \approx \inf_{T_n \in \mathcal{T}_n}\left(\|f-T_n\|_p + n^{-r} \sup_\xi \left\|\left(\frac{\partial}{\partial\xi}\right)^r T_n\right\|_p\right), \tag{5.3$'$}$$

which is equivalent to (5.3). This follows since if the infimum of (5.3)′ is approached by T_n^*, $\|f - T_n^*\|_p \leqslant C\omega^r(f, \frac{1}{n})_p$, and hence with $\alpha = \min(p, 1)$

$$\|T_n - T_n^*\|_p \leqslant \left(\|f-T_n\|_p^\alpha + \left\|f-T_n^*\right\|_p^\alpha\right)^{1/\alpha} \leqslant C\omega^r\left(f, \frac{1}{n}\right)_p.$$

Using the above and the Bernstein inequality, we have

$$n^{-r}\left\|\left(\frac{\partial}{\partial\xi}\right)^r T_n\right\|_p \leqslant n^{-r}\left(\left\|\left(\frac{\partial}{\partial\xi}\right)^r (T_n - T_n^*)\right\|_p^\alpha + \left\|\left(\frac{\partial}{\partial\xi}\right)^r T_n^*\right\|_p^\alpha\right)^{1/\alpha}$$
$$\leqslant C\omega^r\left(f, \frac{1}{n}\right)_p.$$

The extension of [Di-Hr-Iv, Theorem 3.1] from the one-dimensional to the d-dimensional case is completely routine using $(\frac{\partial}{\partial\xi})^r T_n$ instead of $T_n^{(r)}$ there. We now have the assumptions of Theorem 4.4 and hence (2.1) follows. □

Proof of Theorem 3.1. We use Theorem 4.1 to establish (3.3) and (3.4) and Theorems 4.3 and 4.4 to demonstrate (3.2) and (3.1) respectively. We set in Theorem 4.1 $L_{p,w}(D) = L_p[-1,1]$, $\mathcal{A}_n = \Pi_n$ where Π_n is the collection of polynomials of degree $\leqslant n$, and $\beta = 2$. The well-known Nikol'skii-type inequality

$$\|P_n\|_{L_q[-1,1]} \leqslant Cn^{2(\frac{1}{p}-\frac{1}{q})}\|P_n\|_{L_p[-1,1]}, \quad P_n \in \Pi_n, \quad 0 < p \leqslant q \leqslant \infty \tag{5.4}$$

is given in [De-Lo, p. 102 (2.14)]. Therefore, Theorem 4.1 is applicable, and we have (3.3) and (3.4). We set, in addition to the above, $\Omega^\gamma(f,t)_p = \omega_\varphi^r(f,t)_p$ with $\gamma = r$ and $\omega_\varphi^r(f,t)_p$ of (3.5). The Jackson-type result

$$E_n(f)_p \leqslant C\omega_\varphi^r(f,t)_p, \quad 0 < p \leqslant \infty \tag{5.5}$$

was proved in [Di-To, p. 79, Theorem 7.2.1] for $1 \leqslant p \leqslant \infty$ and in [De-Le-Yu] (with the needed (5.6) of [Di-Hr-Iv] see comment there) for $0 < p < 1$. Therefore (4.10) implies (3.2). To prove (3.1) we set in (4.15) $P_\xi(D) = P(D) = \varphi^r(\frac{d}{dx})^r$. Clearly, for $\alpha = \min(p,1)$

$$\omega_\varphi^r(f,t)_q \leqslant \left(\omega_\varphi^r(f-P_n,t)_q^\alpha + \omega_\varphi^r(P_n,t)_q^\alpha\right)^{1/\alpha},$$

and as $\omega_\varphi^r(f-P_n,t)_q \leqslant C\|f-P_n\|_q$ and $\omega_\varphi^r(P_n,t)_q \leqslant Ct^r\|\varphi^r P_n^{(r)}\|_q$ (see [Di-To, Chapter 7] for $1 \leqslant q \leqslant \infty$ and [Di-Hr-Iv, Section 6] for $0 < q < 1$), we have

$$\omega_\varphi^r(f,t)_q \leqslant C_1\left(\|f-P_n\|_q + t^r\|\varphi^r P_n^{(r)}\|_q\right), \tag{5.6}$$

which is (4.11) for our setup. For even r

$$\|\varphi^r P_n^{(r)}\|_{L_q[-1,1]} \leqslant Cn^{2(\frac{1}{p}-\frac{1}{q})}\|\varphi^r P_n^{(r)}\|_{L_p[-1,1]} \tag{5.7}$$

is satisfied since (5.4) is satisfied and $\varphi^r P_n^{(r)} \in \Pi_n$. For odd r we use $p_1 = \frac{p}{2}$, $q_1 = \frac{q}{2}$ and (5.7) follows from

$$\begin{aligned}\left\|\varphi^r P_n\right\|_q^2 &= \|\varphi^{2r}P_n^2\|_{q_1}\\ &\leqslant C_1 n^{2(\frac{1}{p_1}-\frac{1}{q_1})}\|\varphi^{2r}P_n^2\|_{p_1}\\ &\leqslant C_1\left(n^{4(\frac{1}{p}-\frac{1}{q})}\|\varphi^r P_n\|_p^2\right).\end{aligned}$$

We would like to mention that the observation above on (5.7) for odd r is due to D. Leviatan and is simpler than our original proof. To complete the proof we recall (see [Di-Hr-Iv, Theorem 5.1]) that

$$\omega_\varphi^r\left(f, \frac{1}{n}\right)_p \approx \|f - P_n\|_{L_p[-1,1]} + n^{-r}\|\varphi^r P_n^{(r)}\|_{L_p[-1,1]} \tag{5.8}$$

with P_n satisfying $E_n(f)_p = \|f - P_n\|_{L_p[-1,1]}$.

While in Theorem 5.1 of [Di-Hr-Iv] an infimum on all $P_n \in \Pi_n$ is written, this infimum can be dropped in the same manner as was done in the proof of (2.1). We now have all the ingredients of Theorem 4.4 and hence (3.1) is proved with $2t$ instead of t on the right hand side. As

$$\omega_\varphi^r(f, 2t)_p \leqslant C\omega_\varphi^r(f, t)_p, \tag{5.9}$$

which follows from [Di-To] for $1 \leqslant p \leqslant \infty$ and from [Di-Hr-Iv, (5.13)] for $0 < p < 1$, we have the result (3.1) as stated. □

6. Nikol'skii-type inequalities

In earlier sections we used the Nikol'skii inequalities which were given in the literature. In this section, we will make some observations which will help us extend the range of some Nikol'skii-type inequalities and prove some new ones.

It can be observed, as is clear from the proof of some special cases (see [De-Lo, p. 102], [Ne-Wi,Gr-Sa], and others), that the case $0 < p \leqslant 2$, $p < q \leqslant \infty$ follows essentially from the case $p = 2$ and $q = \infty$. We formalize this point in the following theorem and proof which we hope will be helpful as some authors are still squeamish when handling L_p, $0 < p < 1$, which for the Nikol'skii inequality is the easy case.

Theorem 6.1. *Let $\mathcal{A}_\sigma$ (or $\mathcal{A}_n$) be a class of functions on a measurable set $\mathcal{D}$ such that $\mathcal{A}_\sigma \subset L_{2,w}(\mathcal{D})$ and $w(x)$ is a measurable weight function satisfying $w(x) > 0$ a.e. on $\mathcal{D}$. Suppose further that*

$$\|\varphi\|_{L_\infty(\mathcal{D})} \leqslant \gamma(\sigma)^{1/2}\,\|\varphi\|_{L_{2,w}(\mathcal{D})} \quad \text{for all} \quad \varphi \in \mathcal{A}_\sigma. \tag{6.1}$$

Then for $0 < p \leqslant 2$ and $p \leqslant q \leqslant \infty$ we have

$$\|\varphi\|_{L_{q,w}(\mathcal{D})} \leqslant (\gamma(\sigma))^{\frac{1}{p}-\frac{1}{q}}\,\|\varphi\|_{L_{p,w}(\mathcal{D})}. \tag{6.2}$$

Proof. Clearly, $\varphi \in \mathcal{A}_\sigma$ is in L_∞ by (6.1). If $\varphi \in L_{p,w}(\mathcal{D})$, then $\varphi \in L_{q,w}(\mathcal{D})$ for $0 < p < q \leqslant \infty$, while if $\varphi \notin L_p$, (6.2) is trivial. We write

$$\begin{aligned}\|\varphi\|_2 &= \left(\int_{\mathcal{D}} |\varphi|^2 w\,dx\right)^{1/2}\\ &= \left(\int_{\mathcal{D}} \left(|\varphi|^{1-\frac{p}{2}}|\varphi|^{\frac{p}{2}}\right)^2 w\,dx\right)^{1/2}\end{aligned}$$

$$\leqslant \|\varphi\|_\infty^{1-\frac{p}{2}} \|\varphi\|_p^{\frac{p}{2}}$$
$$= \|\varphi\|_\infty \|\varphi\|_\infty^{-\frac{p}{2}} \|\varphi\|_p^{\frac{p}{2}}$$
$$\leqslant \gamma(\sigma)^{1/2} \|\varphi\|_2 \|\varphi\|_\infty^{-\frac{p}{2}} \|\varphi\|_p^{\frac{p}{2}}$$

or

$$\|\varphi\|_\infty \leqslant \gamma(\sigma)^{\frac{1}{p}} \|\varphi\|_p.$$

We complete the proof following the Hölder-inequality for $p < q < \infty$ to obtain

$$\|\varphi\|_q \leqslant \|\varphi\|_\infty^{1-\frac{p}{q}} \|\varphi\|_p^{\frac{p}{q}} \leqslant \gamma(\sigma)^{\frac{1}{p}-\frac{1}{q}} \|\varphi\|_p. \qquad \Box$$

In most cases $\gamma(\sigma) = c\sigma^\alpha$ (or $\gamma(n) = cn^\alpha$), but other functions occur as well. In the above theorem and proof the assumption and the conclusion are about one single σ (or n). However, as the knowledgeable reader understands, we usually make the assumption on a continuous collection of classes $\mathcal{A}_\sigma$ or a sequence of classes $\mathcal{A}_n$, and the conclusion is on these classes.

For $p > 2$ the method traditionally used can be summarized in the following general result.

Theorem 6.2. *Suppose a collection of classes $\mathcal{A}_\sigma \subset L_{2,w}(\mathcal{D})$ (or $\mathcal{A}_n \subset L_{2,w}(\mathcal{D})$), and assume (6.1) is valid for those σ (or n). Suppose further $\varphi \in \mathcal{A}_\sigma$ (or $\varphi \in \mathcal{A}_n$) implies for any integer r, $\varphi^r \in \mathcal{A}_{r\sigma}$ (or $\varphi^r \in \mathcal{A}_{rn}$). Then for $0 < p \leqslant q \leqslant \infty$*

$$\|\varphi\|_{L_{q,w}(\mathcal{D})} \leqslant (\gamma(r\sigma))^{\frac{1}{p}-\frac{1}{q}} \|\varphi\|_{L_{p,w}(\mathcal{D})}, \quad \textit{for} \quad r \geqslant \frac{p}{2}, \quad r \in \boldsymbol{N}. \tag{6.3}$$

We note that if $\gamma(\sigma) = (c\sigma)^\beta$ (or $\gamma(n) = (cn)^\beta$), $\gamma(r\sigma) = (cr\sigma)^\beta$ tends to infinity when p does. We observe that while in examples we know $\varphi \in \mathcal{A}_\sigma$ or $\varphi \in \mathcal{A}_n$ implies $\varphi^r \in \mathcal{A}_{\sigma r}$ or $\varphi^r \in \mathcal{A}_{nr}$, an assumption like $\varphi \in \mathcal{A}_\sigma$ implies $\varphi^r \in \mathcal{A}_{mr\sigma}$ for some fixed m would yield the similar inequality

$$\|\varphi\|_{L_{q,w}(\mathcal{D})} \leqslant \gamma(mr\sigma)^{\frac{1}{p}-\frac{1}{q}} \|\varphi\|_{L_{p,w}(\mathcal{D})} \quad \textit{for} \quad r \geqslant \frac{p}{2}, \quad r \in \boldsymbol{N}. \tag{6.3$'$}$$

Proof. The case $p \leqslant 2$ was already settled in Theorem 6.1. To prove (6.3) for $p > 2$ we choose an integer $r \geqslant \frac{p}{2}$ and write

$$\|\varphi^r\|_2 = \left(\int |\varphi|^{2r} \, w \, dx \right)^{1/2}$$
$$= \left(\int \left(|\varphi|^{r-\frac{p}{2}} |\varphi|^{\frac{p}{2}} \right)^2 w \, dx \right)^{1/2}$$
$$\leqslant \|\varphi\|_\infty^{r-\frac{p}{2}} \|\varphi\|_p^{\frac{p}{2}}$$
$$= \|\varphi\|_\infty^r \|\varphi\|_\infty^{-\frac{p}{2}} \|\varphi\|_p^{\frac{p}{2}}.$$

Using (6.1) for $\varphi^r \in \mathcal{A}_{\sigma r}$, we have

$$\|\varphi\|_\infty^{\frac{p}{2}} \leqslant (\gamma(\sigma r))^{\frac{1}{2}} \|\varphi\|_p^{\frac{p}{2}},$$

and hence for $p \leqslant q \leqslant \infty$ with $r \geqslant \frac{p}{2}$,

$$\|\varphi\|_q \leqslant \gamma(\sigma r)^{\frac{1}{p}-\frac{1}{q}} \|\varphi\|_p. \qquad \square$$

As it turns out in many cases (most of those we know), the inclusion of r in the constant is not necessary. This happens when de la Vallée Poussin-type operators are available.

Definition 6.1. For a collection of classes $\mathcal{A}_\sigma$ where $\sigma \in \boldsymbol{N}$ or $\sigma \in [a, \infty)$ for some $a > 0$, a collection of linear operators V_σ are called delayed means or de la Vallée Poussin-type operators if the following conditions are satisfied:

I. $\|V_\sigma f\|_p \leqslant M\|f\|_p \quad \forall f \in L_{p,w}(\mathcal{D}), \quad 1 \leqslant p \leqslant \infty$,
II. $V_\sigma \varphi = \varphi$ for $\varphi \in A_\sigma$,
III. $V_\sigma f \in \mathcal{A}_{L\sigma}$ for some finite integer L independent of σ.

We can now state and prove a Nikol'skii-type result without resorting to $r \geqslant \frac{p}{2}$ given in (6.3) and to the assumption on φ^r in Theorem 6.2. We write the theorem for $\mathcal{A}_n$, but it is valid for $\mathcal{A}_\sigma$ as well.

Theorem 6.3. *Let $\mathcal{A}_n, \mathcal{A}_{Ln} \in L_{p,w}(\mathcal{D})$ and for both $\mathcal{A}_n$ and $\mathcal{A}_{Ln}$* (6.1) *be satisfied. Suppose also that there exist V_n satisfying* I, II *and* III *of Definition* 6.1 *with the prescribed M and L. Then for* $2 < p \leqslant q \leqslant \infty$

$$\|\varphi\|_{L_{q,w}(\mathcal{D})} \leqslant M\,(\gamma(Ln))^{\frac{1}{p}-\frac{1}{q}} \|\varphi\|_{L_{p,w}(\mathcal{D})}, \quad \varphi \in \mathcal{A}_n \tag{6.4}$$

with M and L of Definition 6.1.

We remark that combining (6.2) and (6.4), we may write for $0 < p \leqslant q \leqslant \infty$ and $\varphi \in \mathcal{A}_n$

$$\|\varphi\|_{L_{q,w}(\mathcal{D})} \leqslant \max(\gamma(n)^{\frac{1}{p}-\frac{1}{q}},\ M\gamma(Ln)^{\frac{1}{p}-\frac{1}{q}})\|\varphi\|_{L_{p,w}(\mathcal{D})}. \tag{6.4$'$}$$

Proof. For $2 < p < q$ we have

$$\|V_n f\|_q \leqslant M\|f\|_q, \quad V_n = T : L_q \to L_q \quad \text{or} \quad \|V_n\|_{q,q} \leqslant M,$$

and using (6.2) for $\mathcal{A}_{Ln}$, we have

$$\|V_n f\|_q \leqslant (\gamma(Ln))^{\frac{1}{2}-\frac{1}{q}} \|V_n f\|_2 \leqslant M\,(\gamma(Ln))^{\frac{1}{2}-\frac{1}{q}} \|f\|_2,$$

$$V_n = T : L_2 \to L_q \quad \text{or} \quad \|V_n\|_{2,q} \leqslant M\,(\gamma(Ln))^{\frac{1}{2}-\frac{1}{q}}.$$

We now use the Riesz–Thorin Theorem for $\frac{1}{p} = \frac{\alpha}{2} + \frac{(1-\alpha)}{q}$, that is, $\frac{1}{p} - \frac{1}{q} = \alpha\left(\frac{1}{2} - \frac{1}{q}\right)$ and obtain

$$\|V_n f\|_q \leqslant M\,(\gamma(Ln))^{\alpha(\frac{1}{2}-\frac{1}{q})}\,\|f\|_p = M\,(\gamma(Ln))^{\frac{1}{p}-\frac{1}{q}}\,\|f\|_p.$$

For $\varphi \in \mathcal{A}_n$ we have $V_n\varphi = \varphi$ and hence (6.4) is satisfied. □

The last theorem (observation) is useful and can be applied in various situations. As an example, we present the following corollary.

Corollary 6.4. *For T_n, a trigonometric polynomial of degree n on $\boldsymbol{T}$ and $0 < p \leqslant q \leqslant \infty$ we have*

$$\|T_n\|_{L_q(\boldsymbol{T})} \leqslant 3n^{\frac{1}{p}-\frac{1}{q}}\,\|T_n\|_{L_p(\boldsymbol{T})}, \quad n \geqslant 1. \tag{6.5}$$

Proof. We recall for $p \leqslant 2$, $p \leqslant q \leqslant \infty$ it is known [De-Lo, p. 102] that $\|T_n\|_q \leqslant \left(\frac{2n+1}{2\pi}\right)^{\frac{1}{p}-\frac{1}{q}}\,\|T_n\|_p$.

We set $V_n f = 2\sigma_{2n} f - \sigma_n f$ (the classical de la Vallée Poussin operator) which satisfies Definition 6.1 with $L = 2$ and $M = 3$. As $\left(\frac{2n+1}{2\pi}\right) \leqslant n$ and $\frac{4n+1}{2\pi} \leqslant n$ for $n \geqslant 1$, (6.4)′ implies (6.5). □

We note that for large p (6.5) is superior to the traditional result [De-Lo, p. 102], that is,

$$\|T_n\|_q \leqslant \left(\frac{2nr+1}{2\pi}\right)^{\frac{1}{p}-\frac{1}{q}}\|T_n\|_p, \quad 0 < p \leqslant q \leqslant \infty, \quad r \geqslant \frac{p}{2} \quad \text{and} \quad r \in \boldsymbol{N}.$$

While in this paper we will not need the improvement over (6.3) given in (6.4), we believe that it is a worthwhile observation and note that it is applicable to trigonometric and algebraic polynomials in d variables, $d \geqslant 1$, to spherical harmonics, and to many other situations.

Perhaps the following generalizations of the [Ne-Wi] result can demonstrate the benefit of Theorems 6.1 and 6.3.

Theorem 6.5. *For a function $G_K(x)$, $x \in \boldsymbol{R}^d$ given by*

$$G_K(x) = \left(\frac{1}{2\pi}\right)^{d/2} \int_K g(\xi) e^{i\xi x}\, d\xi, \tag{6.6}$$

where $g \in L_2(K)$ and K is a measurable set in $\boldsymbol{R}^d$, we have

$$\|G_K\|_{L_q(\boldsymbol{R}^d)} \leqslant \left(\frac{m(K)}{(2\pi)^d}\right)^{\frac{1}{p}-\frac{1}{q}}\|G_K\|_{L_p(\boldsymbol{R}^d)}, \quad 0 < p \leqslant 2, \quad p \leqslant q \leqslant \infty. \tag{6.7}$$

If in addition $K \subset I_\sigma = [-\sigma, \sigma] \times \cdots \times [-\sigma, \sigma]$, we have

$$\|G_K\|_{L_q(\boldsymbol{R}^d)} \leqslant (c\sigma^d)^{\frac{1}{p}-\frac{1}{q}}\|G_K\|_{L_p(\boldsymbol{R}^d)}, \quad 0 < p \leqslant q \leqslant \infty \tag{6.8}$$

with c independent of p and q.

In an analogue of Theorem 6.5 given in [Ne-Wi] it is assumed that K is compact, convex and symmetric. We note that being compact is not needed for (6.7), and being convex and symmetric is not needed for (6.8). Also, the constant situation in (6.8) is better than in [Ne-Wi] as we do not resort to $G_K \in \mathcal{A}_\sigma$ implies $G_K^r \in \mathcal{A}_{r\sigma}$ but use Theorem 6.3 instead. However, if the constant was of no concern, we could have deduced (6.8) from [Ne-Wi].

Proof. Using the Cauchy–Schwartz inequality, we have

$$\|G_K\|_{L_\infty(\boldsymbol{R}^d)} \leqslant \left(\frac{m(K)}{2\pi}\right)^{1/2} \|g\|_{L_2(K)}.$$

Defining $g(\xi) = 0$ for $\xi \notin K$, we have

$$\|g\|_{L_2(K)} = \|g\|_{L_2(\boldsymbol{R}^d)} = \|G_K\|_{L_2(\boldsymbol{R}^d)},$$

and hence we have an inequality of type (6.1) which, using Theorem 6.1, implies (6.7). We need to prove (6.8) only for $p > 2$ as it is weaker than (6.7) for $p \leqslant 2$.

We set $H_\sigma(x_i) = \frac{1}{2\pi\sigma}\left(\frac{\sin\frac{\sigma x_i}{2}}{x_i/2}\right)^2$ which satisfy $\int_{-\infty}^{\infty} H_\sigma(x_i)\,dx_i = 1$, and we note that

$$\left\|\int_{-\infty}^{\infty} H_\sigma(x_i) f(y-x)\,dx_i\right\|_B \leqslant \|f\|_B$$

(where $x, y \in \boldsymbol{R}^d$, $x = (x_1, \dots, x_i, \dots x_d)$) holds for $B = L_1(\boldsymbol{R}^d)$ and $B = L_\infty(\boldsymbol{R}^d)$, and hence it holds for $B = L_p(\boldsymbol{R}^d)$ for all $1 \leqslant p \leqslant \infty$. Therefore,

$$V_\sigma f(y) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{i=1}^{d} (2H_{2\sigma}(x_i) - H_\sigma(x_i))\, f(y-x)dx_1 \dots dx_d$$

satisfies

$$\|V_\sigma f\|_{L_p(R^d)} \leqslant 3^d \|f\|_{L_p(R^d)} \quad \text{for} \quad 1 \leqslant p \leqslant \infty.$$

For G_K given by (6.6) where $K \subset I_\sigma$ we have $V_\sigma G_K = G_K$. For $f \in L_2(\boldsymbol{R}^d)$, $V_\sigma f$ is a Fourier transform of a function in L_2 supported by $I_{2\sigma}$. As $V_\sigma f \in L_2(\boldsymbol{R}^d)$, and using (6.7) with $p = 2$, $q > 2$ and $K = I_{2\sigma}$, we have

$$\|V_\sigma f\|_{L_q(R^d)} \leqslant \left(\frac{(2\sigma)^d}{(2\pi)^d}\right)^{\frac{1}{2}-\frac{1}{q}} \|V_\sigma f\|_{L_2(R^d)}.$$

From the above consideration and Theorem 6.3, we have (6.8). □

For polynomials with Jacobi weights on the cube we have the following result.

Theorem 6.6. *Suppose* $w = w_{\alpha,\beta}(x) = \prod_{i=1}^{d} w_{\alpha_i\beta_i}(x_i)$ *for* $x \in I_d = [-1,1]\times\cdots\times[-1,1]$ *where* $w_{\alpha_i,\beta_i}(x_i) = (1-x_i)^{\alpha_i}(1+x_i)^{\beta_i}$, $\alpha_i > -1, \beta_i > -1, \alpha_i+\beta_i > -1, \alpha = (\alpha_1, \dots, \alpha_d)$,

$\beta = (\beta_1, \ldots, \beta_d)$ and $x = (x_1, \ldots, x_d)$. Then for $0 < p \leqslant q \leqslant \infty$ we have

$$\|P_n\|_{L_{q,w}(I_d)} \leqslant C n^{\gamma(\frac{1}{p}-\frac{1}{q})} \|P_n\|_{L_{p,w}(I_d)}, \tag{6.9}$$

where $\gamma = \sum_{i=1}^{d} \max\left(2 + 2\max(\alpha_i, \beta_i), 1\right)$ and P_n a polynomial of total degree n.

In particular, if $w(x) = 1$, we have for $0 < p \leqslant q \leqslant \infty$

$$\|P_n\|_{L_q(I_d)} \leqslant C n^{2d(\frac{1}{p}-\frac{1}{q})} \|P_n\|_{L_p(I_d)}. \tag{6.9$'$}$$

Proof. Using Theorems 6.1 and 6.2, it is sufficient to prove (6.9) with $q = \infty$ and $p = 2$. (We could have used Theorem 6.3 to improve the constant, but the construction of V_n would lead us too far from the topic.) A polynomial of total degree $\leqslant n$ is of degree $\leqslant n$ in each variable, and hence

$$\begin{aligned} &P_n(y) \\ &= \int_{-1}^{1} \cdots \int_{-1}^{1} P_n(x) w_{\alpha,\beta}(x) \left[\prod_{i=1}^{d} \sum_{k=0}^{n} Q_k^{(\alpha_i,\beta_i)}(x_i) Q_k^{(\alpha_i,\beta_i)}(y_i) \right] dx_1 \ldots dx_d, \end{aligned}$$

where $Q^{(\alpha_i,\beta_i)}(x_i)$ is the orthonormal system of polynomials on $[-1, 1]$ with weight $w_{\alpha_i,\beta_i}(x_i)$. Therefore, using the Cauchy–Schwartz inequality, we have

$$\begin{aligned} \|P_n\|_{L_{\infty,w_{\alpha,\beta}}[I_d]} &= \|P_n\|_{L_\infty[I_d]} \\ &\leqslant \|P_n\|_{L_{2,w_{\alpha,\beta}}[I_d]} \sup_{\substack{-1 \leqslant y_i \leqslant 1 \\ 1 \leqslant i \leqslant d}} \left[\prod_{i=1}^{d} \left(\sum_{k=0}^{n} \left(Q_k^{(\alpha_i,\beta_i)}(y_i) \right)^2 \right) \right]^{1/2} \\ &\leqslant \|P_n\|_{L_{2,w_{\alpha,\beta}}[I_d]} \left(\prod_{i=1}^{d} \sum_{k=0}^{n} \sup_{-1 \leqslant y_i \leqslant 1} \left(Q_k^{(\alpha_i,\beta_i)}(y_i) \right)^2 \right)^{1/2}. \end{aligned}$$

We use Szëgo estimates of $P_k^{(\alpha,\beta)}(\xi)$, $\xi \in [-1, 1]$, [Sz, (7.32.2), p. 166]

$$\max |P_k^{(\alpha,\beta)}(\xi)| \leqslant \begin{cases} C_1 k^{\lambda} & \lambda = \max(\alpha, \beta) \geqslant -\frac{1}{2} \\ C_1 k^{-1/2} & \lambda = \max(\alpha, \beta) < -\frac{1}{2} \end{cases} ; \quad k \geqslant 1$$

and recall the relation between $P_k^{(\alpha,\beta)}(\xi)$ and $Q_k^{(\alpha,\beta)}(\xi)$ which follows from [Sz, (4.3.3), p. 68],

$$|Q_k^{(\alpha,\beta)}(\xi)| \leqslant C_2 k^{1/2} |P_k^{(\alpha,\beta)}(\xi)|, \quad k \geqslant 1,$$

where both C_1 and C_2 are independent of k. Therefore, setting $\xi = y_i$,

$$\sum_{k=0}^{n} \sup_{-1 \leqslant y_1 \leqslant 1} \left(Q_k^{(\alpha,\beta)}(y_i) \right)^2 \leqslant \begin{cases} C_3 \left(1 + \sum_{k=1}^{n} k^{2\lambda+1} \right) \leqslant C_4 n^{2\lambda+2}, & \lambda \geqslant -\frac{1}{2} \\ C_3 \left(1 + \sum_{k=1}^{n} 1 \right) \leqslant C_4 n, & \lambda < -\frac{1}{2} \end{cases},$$

which implies (6.9). $\square$

Partial results of the above theorem were known (see [Da-Ra]). We will use parts of Theorem 6.6 in some of the following sections.

Remark 6.7. The power γ in (6.9) is sharp at least when $\max(\alpha_i, \beta_i) \geqslant -\frac{1}{2}$, as can be seen when $q = \infty$, $p = 2$ and $P_n(x) = \prod_{i=1}^{d} \sum_{k=0}^{n} \varepsilon(k,i) Q_k^{(\alpha_i,\beta_i)}(x)$ with $\varepsilon(k,i) = 1$ if $\alpha_i \geqslant \beta_i$ and $\varepsilon(k,i) = (-1)^k$ if $\beta_i > \alpha_i$. (The fact that $P_n(x)$ is of total degree nd does not make a difference.)

7. Ul'yanov-type inequality on *R*

For the Ul'yanov result on $\boldsymbol{R}$ (without weight) we use the collection of linear spaces $\mathcal{A}_\sigma$ defined by $G_\sigma \in \mathcal{A}_\sigma$ if

$$G_\sigma(x) = \frac{1}{\sqrt{2\pi}} \int_{-\sigma}^{\sigma} g(\xi) e^{i\xi x}\, d\xi, \quad g \in L_2, \tag{7.1}$$

that is, the collection of exponential functions of type σ. The rate of best approximation in $L_p(\boldsymbol{R})$ is given by

$$E_\sigma(f)_p = \inf\{\|f - G_\sigma\|_{L_p(R)}; G_\sigma \in \mathcal{A}_\sigma\}. \tag{7.2}$$

The moduli of smoothness are defined as usual by

$$\omega^r(f,t)_p = \sup_{|h|<t} \|\Delta_h^r f\|_{L_p(\boldsymbol{R})}, \quad \Delta_h f(x) = f(x+h) - f(x),$$
$$\Delta_h^r = \Delta_h(\Delta_h^{r-1}). \tag{7.3}$$

The inequalities are given in the following theorem.

Theorem 7.1. *For $f \in L_p(\boldsymbol{R})$, $0 < p < q \leqslant \infty$, we have*

$$\|f\|_{L_q(\boldsymbol{R})} \leqslant C \left[\left\{ \int_1^\infty \sigma^{q_1\theta} E_\sigma(f)_p^{q_1} \frac{d\sigma}{\sigma} \right\}^{1/q_1} + \|f\|_{L_p(\boldsymbol{R})} \right], \tag{7.4}$$

$$E_\sigma(f)_q \leqslant C \left\{ \int_\sigma^\infty \eta^{q_1\theta} E_\eta(f)_p^{q_1} \frac{d\eta}{\eta} \right\}^{1/q_1}, \tag{7.5}$$

$$\|f\|_{L_q(\boldsymbol{R})} \leqslant C \left[\left\{ \int_0^1 \left(u^{-\theta}\omega^r(f,u)_p\right)^{q_1} \frac{du}{u} \right\}^{1/q_1} + \|f\|_{L_p(\boldsymbol{R})} \right] \tag{7.6}$$

and

$$\omega^r(f,t)_q \leqslant C \left\{ \int_0^t \left(u^{-\theta}\omega^r(f,u)_p\right)^{q_1} \frac{du}{u} \right\}^{1/q_1}, \tag{7.7}$$

where $q_1 = \begin{cases} q, & q < \infty \\ 1, & q = \infty \end{cases}$, $\theta = \left(\frac{1}{p} - \frac{1}{q}\right)$, *$E_\sigma(f)_p$ is given by* (7.2) *and $\omega^r(f,u)_p$ is given by* (7.3).

Proof. The Nikol'skii inequality

$$\|G_\sigma\|_{L_q(\mathbf{R})} \leqslant C\sigma^{\frac{1}{p}-\frac{1}{q}} \|G_\sigma\|_{L_p(\mathbf{R})}, \quad 0 < p \leqslant q \leqslant \infty, \quad G_\sigma \in A_\sigma \tag{7.8}$$

is well-known, and in fact for $p \geqslant 1$ goes back to Nikol'skii (see [Ni, Ne-Wi]). Therefore, Theorem 4.1 implies (7.4) and (7.5). The Jackson inequality was proved by Taberski [Ta] for $0 < p < 1$ and was known earlier for $1 \leqslant p \leqslant \infty$. This implies (7.6) using Theorem 4.3. The realization result [Di-Hr-Iv, Section 4] was given by

$$\omega^r\left(f, \frac{1}{\sigma}\right)_p \approx \inf_{G_\sigma \in \mathcal{A}_\sigma} \left(\|f - G_\sigma\|_p + \sigma^{-r}\|G_\sigma^{(r)}\|_p\right), \quad 0 < p \leqslant \infty, \tag{7.9}$$

and using the argument deriving (5.3) from (5.3)′ and the Jackson-type estimate here, we derive

$$\omega^r\left(f, \frac{1}{\sigma}\right)_p \approx \|f - G_\sigma\|_p + \sigma^{-r}\|G_\sigma^{(r)}\|_p, \tag{7.9$'$}$$

where $\|G_\sigma - f\|_p = E_\sigma(f)_p$ (or $\|G_\sigma - f\|_p \leqslant AE_\sigma(f)_p$).

We now use (7.9) for L_q with G_σ given by $\|G_\sigma - f\|_{L_p} = E_\sigma(f)_p$ to yield the appropriate form of (4.11) here. We then use the Jackson estimate

$$E_\sigma(f)_p \leqslant C\omega^r\left(f, \frac{1}{\sigma}\right)_p,$$

which is of the form of (4.9) here. Using (7.9)′, we obtain

$$\sigma^{-r}\left\|G_\sigma^{(r)}\right\|_p \leqslant C\omega^r\left(f, \frac{1}{\sigma}\right)_p \quad \text{for} \quad \|G_\sigma - f\|_p = E_\sigma(f)_p,$$

which is what we need for (4.12). This completes the assembly of all ingredients needed for the proof of (7.7). □

For the multidimensional analogue we do not have the appropriate Jackson and realization results for the range $0 < p \leqslant \infty$. However, as a corollary of Theorem 4.1, we can state and prove the following theorem. (We can also prove a partial analogue of (7.6) and (7.7) for $1 \leqslant p < q \leqslant \infty$).

Theorem 7.2. *Suppose $f \in L_p(\mathbf{R}^d)$ and $0 < p < q \leqslant \infty$. Then*

$$\|f\|_{L_q(\mathbf{R}^d)} \leqslant C\left[\left\{\int_1^\infty \sigma^{q_1\theta} E_\sigma(f)_p^{q_1} \frac{d\sigma}{\sigma}\right\}^{1/q_1} + \|f\|_{L_p(\mathbf{R}^d)}\right] \tag{7.10}$$

and

$$E_\sigma(f)_q \leqslant C\left\{\int_\sigma^\infty \eta^{q_1\theta} E_\eta(f)_p^{q_1} \frac{d\eta}{\eta}\right\}^{1/q_1} \tag{7.11}$$

where $q_1 = \begin{cases} q, & q < \infty \\ 1, & q = \infty \end{cases}, \quad \theta = d\left(\frac{1}{p} - \frac{1}{q}\right),$

$$E_\sigma(f)_{L_p(\boldsymbol{R}^d)} = \inf \left(\|f - G_\sigma\|_{L_p(\boldsymbol{R}^d)};\ G_\sigma \in \mathcal{A}_\sigma \right),$$

and $G_\sigma \in \mathcal{A}_\sigma$ if

$$G_\sigma(x) = \left(\frac{1}{\sqrt{2\pi}}\right)^d \int_{-\sigma}^{\sigma} \cdots \int_{-\sigma}^{\sigma} e^{ix\cdot\xi} g(\xi) d\xi_1 \ldots d\xi_d,$$
$$x, \xi \in \boldsymbol{R}^d \quad \textit{and} \quad g \in L_2.$$

Proof. The results follow immediately from Theorem 4.1 and the Nikol'skii inequality [Ti,A, p. 235 [34]] and [Ne-Wi]. □

8. Approximation by polynomials on simple polytopes

A region $S \subset \boldsymbol{R}^d$ is a simple polytope if it is a polytope (convex hull of finitely many points) which has an interior point and whose vertices are connected to adjacent vertices by exactly d edges. The best rate of approximation is given by

$$E_{n,S}(f)_p = \inf \left(\|f - P_n\|_{L_p(S)};\ P_n \in \Pi_n \right), \tag{8.1}$$

where Π_n is the collection of polynomials of total degree $\leqslant n$. The moduli of smoothness we use is

$$\omega_S^r(f,t)_p = \sup \left(\|\Delta^r_{h\varphi_\xi \xi} f\|_{L_p(S)};\ |h| \leqslant t,\ |\xi| = 1,\ \xi \in E_S \right), \quad r \in \boldsymbol{N}, \tag{8.2}$$

where E_S is the set of edges of S,

$$\varphi_\xi(x)^2 = \widetilde{d}_S(x,\xi) = \inf_{\substack{x+\lambda\xi \in S \\ \lambda \geqslant 0}} d(x, x+\lambda\xi) \inf_{\substack{x-\lambda\xi \in S \\ \lambda \geqslant 0}} d(x, x-\lambda\xi), \tag{8.3}$$

$d(x, y)$ is the Euclidean distance between x and y and

$$\Delta_u^r f(x) = \begin{cases} \sum_{k=0}^{r} \binom{r}{k} (-1)^k f\left(x + \left(\frac{r}{2} - k\right) u\right) & \text{for} \quad x \pm \frac{r}{2} u \in S, \\ 0 & \text{otherwise.} \end{cases}$$

Best approximation as well as moduli of smoothness for different $L_p(S)$ are related by the following theorem.

Theorem 8.1. *Suppose S is a simple polytope,* $f \in L_p(S)$ *and* $0 < p < q \leqslant \infty$. *Then*

$$\|f\|_{L_q(S)} \leqslant C \left[\left\{ \sum_{k=1}^{\infty} k^{q_1\theta - 1} E_{k,S}(f)_p^{q_1} \right\}^{1/q_1} + \|f\|_{L_p(S)} \right], \tag{8.4}$$

$$E_{n,S}(f)_q \leqslant C\left\{\sum_{k=n}^{\infty} k^{q_1\theta-1}E_{k,S}(f)_p^{q_1}\right\}^{1/q_1}, \tag{8.5}$$

$$\|f\|_{L_q(S)} \leqslant C\left[\left\{\int_0^1 \left(u^{-\theta}\omega_S^r(f,u)_p\right)^{q_1}\frac{du}{u}\right\}^{1/q_1} + \|f\|_{L_p(S)}\right] \tag{8.6}$$

and

$$\omega_S^r(f,t)_q \leqslant C\left\{\int_0^t \left(u^{-\theta}\omega_S^r(f,u)_p\right)^{q_1}\frac{du}{u}\right\}^{1/q_1}, \tag{8.7}$$

where $q_1 = \begin{cases} q, & q<\infty \\ 1, & q=\infty \end{cases}$, $\theta = 2d\left(\frac{1}{p}-\frac{1}{q}\right)$ *and where* $E_{n,S}(f)_p$ *and* $\omega_S^r(f,t)_p$ *are given by* (8.1) *and* (8.2), *respectively*.

For the proof of Theorem 8.1 we have most of the necessary ingredients. However, an essential inequality, that is, the appropriate Nikol'skii-type inequality is missing and will be given in the following lemma.

Lemma 8.2. *For a simple polytope S*, $S \subset \boldsymbol{R}^d$ *and* $0 < p \leqslant q \leqslant \infty$ *we have*

$$\|P_n\|_{L_q(S)} \leqslant Cn^{2d(\frac{1}{p}-\frac{1}{q})}\|P_n\|_{L_p(S)}, \tag{8.8}$$

where P_n *is a polynomial of total degree n and C depends on S and p but not on n or* P_n.

Proof of Lemma 8.2. For the Box $B=[-1,1]\times\cdots\times[-1,1]$ the inequality of our lemma is (6.9)′. For an affine transformation of the Box B_A the result is still valid, and we have

$$\|P_n\|_{L_q(B_A)} \leqslant C|J(A)|^{\frac{1}{p}-\frac{1}{q}} n^{2d(\frac{1}{p}-\frac{1}{q})}\|P_n\|_{L_p(B_A)},$$

where $J(A)$ is the Jacobian of the affine transformation. In case $|J(A)| \leqslant 1$ we replace $|J(A)|^{\frac{1}{p}-\frac{1}{q}}$ by 1, and otherwise by $|J(A)|^{\frac{1}{p}}$, and hence our constant depends on C of (6.9)′, on p and on B_A. A simple polytope can be covered by a finite number of $B_{A_i} \subset S$, that is $S \subset \bigcup_{i=1}^{L} B_{A_i}$ and hence

$$\begin{aligned}
\|P_n\|_{L_q(S)} &\leqslant \sum_{i=1}^{L}\|P_n\|_{L_q(B_{A_i})} \\
&\leqslant Cn^{2d(\frac{1}{p}-\frac{1}{q})}\sum_{i=1}^{L}|J(A_i)|^{\frac{1}{p}-\frac{1}{q}}\|P_n\|_{L_p(B_{A_i})} \\
&\leqslant C_1 n^{2d(\frac{1}{p}-\frac{1}{q})}\max_{1\leqslant i\leqslant L}|J(A_i)|^{\frac{1}{p}-\frac{1}{q}}\|P_n\|_{L_p(B_{A_i})} \\
&\leqslant C_1 n^{2d(\frac{1}{p}-\frac{1}{q})}\|P_n\|_{L_p(S)}. \qquad \square
\end{aligned}$$

Proof of Theorem 8.1. Using the definitions, Lemma 8.2 and Theorem 4.1, we obtain (8.4) and (8.5). We now use (4.1) and (4.3) of [Di,I, Theorem 4.1, p. 252] to derive the Jackson-type inequality

$$E_{n,S}(f)_p \leqslant C\omega_S^r(f,t)_p$$

which, using Theorem 4.3, implies (8.6). We use (4.2) of [Di,I, Theorem 4.1] with L_q to get the appropriate (4.11) with P_n the best $L_p(S)$ approximant to *f*. We note that here we use in (4.15)

$$\Phi(P_n)_q = \sup_{\xi\in\Xi} \|P_\xi(D)P_n\|_q = \sup_{\xi\in E_S} \left\| \varphi_\xi^r \left(\frac{\partial}{\partial\xi}\right)^r P_n \right\|_q$$

with $\Xi = E_S$. We recall that $\varphi_\xi(x)^2$ and $\left(\frac{\partial}{\partial\xi}\right)^r P_n(x)$ are polynomials (for any ξ), and hence we use the Nikol'skii inequality of Lemma 8.2 with $p_1 = \frac{p}{2}$ and $q_1 = \frac{q}{2}$ to get

$$\begin{aligned}\left\| \varphi_\xi^r \left(\frac{\partial}{\partial\xi}\right)^r P_n \right\|_{L_q(S)}^2 &= \left\| \varphi_\xi^{2r} \left(\left(\frac{\partial}{\partial\xi}\right)^r P_n\right)^2 \right\|_{L_{q_1}(S)} \\ &\leqslant C_1 n^{2d\left(\frac{1}{p_1}-\frac{1}{q_1}\right)} \left\| \varphi_\xi^{2r} \left(\left(\frac{\partial}{\partial\xi}\right)^r P_n\right)^2 \right\|_{L_{p_1}(S)} \\ &\leqslant C_1 n^{4d\left(\frac{1}{p}-\frac{1}{q}\right)} \left\| \varphi_\xi^r \left(\frac{\partial}{\partial\xi}\right)^r P_n \right\|_{L_p(S)}^2 .\end{aligned}$$

The above implies

$$\sup_{\xi\in E_S} \left\| \varphi_\xi^r \left(\frac{\partial}{\partial\xi}\right)^r P_n \right\|_{L_q(S)} \leqslant Cn^{2d\left(\frac{1}{p}-\frac{1}{q}\right)} \sup_{\xi\in E_S} \left\| \varphi_\xi^r \left(\frac{\partial}{\partial\xi}\right)^r P_n \right\|_{L_p(S)},$$

which is the appropriate form of (4.12) here. To obtain the inequality

$$n^{-r} \sup_{\xi\in E_S} \left\| \varphi_\xi^r \left(\frac{\partial}{\partial\xi}\right)^r P_n \right\|_{L_p(S)} \leqslant C\omega_S^r\left(f,\frac{1}{n}\right)_p$$

for P_n satisfying $\|f-P_n\|_p = E_n(f)_p$ we use (4.4) of [Di,I, Theorem 4.1]. This implies (8.7) with $2t$ on the right, which can be restored to t using again (4.3) of [Di,I]. □

9. Ul'yanov-type inequalities, Freud's weights

Freud's weights are given by

$$w_Q(s) = w(x) = \exp(-Q(x))$$

with some conditions on $Q(x)$. There are many different versions of these conditions, as can be seen in [Di-Lu, p. 101, Definition 1.1], [Di-To, p. 101, Definition 11.2.1], [Le-Lu,

p. 10, Definition 1.2] and [Mh, p. 47, Definition 3.1.1]. All these definitions have in common one thing: they are based on the prototype

$$w(x) = w_\alpha(x) = \exp(-|x|^\alpha), \quad \alpha > 1. \tag{9.1}$$

As different results which we need here are based on different definitions, and as dealing with Freud's weights is not the main subject here, we deal only with $w_\alpha(x)$. This simplifies the description of θ of Theorems 4.1, 4.3 and 4.4 as well as guarantees that all the ingredients needed for the use of these theorems are valid. That is, for these weights, the Nikol'skii and Jackson-type inequalities as well as the realization result were proved earlier.

We define the moduli of smoothness following [Di-Lu, (1.11), (1.15) and (1.16)] by

$$\begin{aligned}\omega_\alpha^r(f,t)_p &= \omega^r(f, w_\alpha, t)_p \\ &\equiv \sup_{0<h\leqslant t} \|w_\alpha \Delta_h^r f\|_{L_p[x;|x|\leqslant h^{1/1-\alpha}]} \\ &\quad + \inf_{P\in\Pi_{r-1}} \|(f-P)w_\alpha\|_{L_p[x;|x|\geqslant t^{1/1-\alpha}]}, \quad r\in\mathbf{N}.\end{aligned} \tag{9.2}$$

In [Di-To, 11.2.2, p. 182] somewhat different moduli of smoothness are defined for $1\leqslant p\leqslant\infty$; however, we need (9.2) as we want the moduli to be defined for $0<p<1$ as well. The best weighted rate of approximation is given by

$$E_n(f)_{\alpha,p} = \inf\left(\|w_\alpha(f-P_n)\|_{L_p(\mathbf{R})};\ P_n\in\Pi_n\right), \tag{9.3}$$

which is a somewhat different expression than (4.4), and as a result of it, we will have to be careful when proving the theorem of this section.

Theorem 9.1. *For $0<p<q\leqslant\infty$ and for w_α, $E_n(f)_{\alpha,p}$ and $\omega_\alpha^r(f,t)_p$ given by (9.1), (9.3) and (9.2) respectively we have*

$$\|w_\alpha f\|_{L_q(\mathbf{R})} \leqslant C\left[\left\{\sum_{k=1}^\infty k^{q_1\theta-1}E_k(f)_{\alpha,p}^{q_1}\right\}^{1/q_1} + \|w_\alpha f\|_{L_p(\mathbf{R})}\right], \tag{9.4}$$

$$E_n(f)_{\alpha,q} \leqslant C\left\{\sum_{k=n}^\infty k^{q_1\theta-1}E_k(f)_{\alpha,p}^{q_1}\right\}^{1/q_1}, \tag{9.5}$$

$$\|w_\alpha f\|_{L_q(\mathbf{R})} \leqslant C\left[\left\{\int_0^1 \left(u^{-\eta}\omega_\alpha^r(f,u)_p\right)^{q_1}\frac{du}{u}\right\}^{1/q_1} + \|w_\alpha f\|_{L_p(\mathbf{R})}\right] \tag{9.6}$$

and

$$\omega_\alpha^r(f,t)_q \leqslant C\left\{\int_0^t \left(u^{-\eta}\omega_\alpha^r(f,u)_p\right)^{q_1}\frac{du}{u}\right\}^{1/q_1}, \tag{9.7}$$

where $q_1 = \begin{cases} q & q<\infty \\ 1 & q=\infty\end{cases}$, $\theta = \frac{\alpha-1}{\alpha}\left(\frac{1}{p}-\frac{1}{q}\right)$ *and* $\eta = \frac{1}{p}-\frac{1}{q}$.

Proof. We use the Nikol'skii-type inequality proved by Nevai and Totik [Ne-To], that is

$$\|P_n w_\alpha\|_{L_p(\mathbf{R})} \leqslant C_{p,q} n^{\frac{\alpha-1}{\alpha}(\frac{1}{p}-\frac{1}{q})} \|P_n w_\alpha\|_{L_q(\mathbf{R})} \tag{9.8}$$

for $P_n \in \Pi_n$, $0 < p \leqslant q \leqslant \infty$ and $\alpha > 1$. As in the Nikol'skii-type inequality used, not $f \in L_{p,w}(\mathbf{R})$ but $w_\alpha f \in L_p(\mathbf{R})$ (or $L_q(\mathbf{R})$), we can follow Lemma 4.2 where the only place of change is when the Nikol'skii inequality is utilized and we have instead of (4.8)

$$\left\| \sum_{\ell=1}^{m} \left(P_{n2^\ell} - P_{n2^{\ell-1}}\right) w_\alpha \right\|_{L_q(\mathbf{R})} \leqslant C \left(\sum_{\ell=1}^{m} \left(\left(n2^\ell\right)^{\frac{\alpha-1}{\alpha}\left(\frac{1}{p}-\frac{1}{q}\right)} E_{n2^{\ell-1}}(f)_{\alpha,p} \right)^{q_1} \right)^{1/q_1}$$

where P_n is best (or near best) approximate to *f*.

Following now the proof of Theorem 4.1, we have (9.4) and (9.5). We now use the Jackson inequality, which is part of [Di-Lu, Theorem 1.2], together with [Di-Lu, Theorem 1.4, (1.24)] to obtain

$$E_n(f)_{\alpha,p} \leqslant C\omega_\alpha^r \left(f, n^{\frac{1}{\alpha}-1}\right)_p. \tag{9.9}$$

We now obtain (9.6) when we write

$$\begin{aligned}
&\left\{ \sum_{\ell=1}^{\infty} \left(\left(2^\ell\right)^{\frac{\alpha-1}{\alpha}\left(\frac{1}{p}-\frac{1}{q}\right)} E_{2^{\ell-1}}(f)_{\alpha,p} \right)^{q_1} \right\}^{1/q_1} \\
&\quad \leqslant C \left\{ \sum_{\ell=1}^{\infty} \left(\left(2^\ell\right)^{\frac{\alpha-1}{\alpha}\left(\frac{1}{p}-\frac{1}{q}\right)} \omega_\alpha^r \left(f, 2^{\ell(\frac{1}{\alpha}-1)}\right)_p \right)^{q_1} \right\}^{1/q_1} \\
&\quad \leqslant C_1 \left\{ \int_0^1 \left(v^{-\frac{\alpha-1}{\alpha}\left(\frac{1}{p}-\frac{1}{q}\right)} \omega_\alpha^r \left(f, v^{\frac{\alpha-1}{\alpha}}\right)_p \right)^{q_1} \frac{dv}{v} \right\}^{1/q_1} \\
&\quad \leqslant C_2 \left\{ \int_0^1 \left(u^{-(\frac{1}{p}-\frac{1}{q})} \omega_\alpha^r(f; u)_p \right)^{q_1} \frac{du}{u} \right\}^{1/q_1}.
\end{aligned}$$

To prove (9.7) we use [Di-Lu, Theorem 1.4] to write for all $0 < p \leqslant \infty$ (including *q*)

$$\omega_\alpha^r \left(f, n^{\frac{1}{\alpha}-1}\right)_p \approx \inf_{P_n \in \Pi_n} \left(\|(f - P_n)w_\alpha\|_{L_p(\mathbf{R})} + \left(n^{\frac{1}{\alpha}-1}\right)^r \|P_n^{(r)} w_\alpha\|_{L_p(\mathbf{R})} \right) \tag{9.10}$$

as $\omega_\alpha^r(f,t)_p$ is $\omega_{r,p}(f, w_\alpha, t)$ of [Di-Lu], and on the right hand side of (9.10) we have $\overline{K}_{r,p}(f, w_\alpha, t^r)$ with $t = n^{\frac{1}{\alpha}-1}$ (recall (1.23) of [Di-Lu]). Therefore, for P_n satisfying

$$\|(P_n - f)w_\alpha\|_{L_p(\mathbf{R})} = E_n(f)_{\alpha,p} \tag{9.11}$$

we have

$$\omega_\alpha^r \left(f, n^{\frac{1}{\alpha}-1}\right)_q \leqslant C \left(\|(f - P_n)w_\alpha\|_{L_q(\mathbf{R})} + \left(n^{\frac{1}{\alpha}-1}\right)^r \|P_n^{(r)} w_\alpha\|_{L_q(\mathbf{R})} \right), \tag{9.12}$$

which is the needed analogue of (4.11). As $P_n^{(r)}$ is also a polynomial, the analogue of (4.12) is the Nikol'skii inequality proved by Nevai and Totik (see [Ne-To]). We now use (9.10) (for p this time) and recall that the same argument that allowed us to change from (5.3)′ to (5.3) implies here (since (9.9) was established) for P_n satisfying (9.11)

$$\omega_\alpha^r\left(f, n^{\frac{1}{\alpha}-1}\right)_p \approx \|(f-P_n)w_\alpha\|_{L_p(\boldsymbol{R})} + \left(n^{\frac{1}{\alpha}-1}\right)^r \|P_n^{(r)} w_\alpha\|_{L_p(\boldsymbol{R})}, \tag{9.13}$$

and hence

$$\left(n^{\frac{1}{\alpha}-1}\right)^r \|P_n^{(r)} w_\alpha\|_{L_p(\boldsymbol{R})} \leqslant C\omega_\alpha^r\left(f, n^{\frac{1}{\alpha}-1}\right)_p, \tag{9.14}$$

which takes the place of (4.13). Therefore, we have the ingredients prescribed in Theorem 4.4 and we obtain (9.7). □

10. Smoothness on the sphere and spherical harmonics

The unit sphere $S_{d-1} \subset \boldsymbol{R}^d$ is given by

$$S_{d-1} = \left\{x \in \boldsymbol{R}^d;\ |x|^2 = x_1^2 + \cdots + x_d^2 = 1\right\}.$$

The eigenspace of spherical harmonics of degree k is given by

$$H_k = \{\varphi : \widetilde{\Delta}\varphi = -k(k+d-1)\varphi\}, \quad \widetilde{\Delta} f(x) = \Delta f\left(\frac{x}{|x|}\right), \tag{10.1}$$

where $\widetilde{\Delta}$ is the Laplace–Beltrami operator and Δ is the Laplacian. The rate of best approximation is given by

$$E_n(f)_p = \inf\left(\|f-\varphi\|_{L_p(S_{d-1})} : \varphi \in \operatorname{span}\left\{\bigcup_{k=0}^n H_k\right\}\right). \tag{10.2}$$

We have the following result:

Theorem 10.1. *Suppose* $f \in L_p(S_{d-1})$, $0 < p < q \leqslant \infty$. *Then*

$$\|f\|_{L_q(S_{d-1})} \leqslant C\left[\left\{\sum_{k=1}^\infty k^{\theta q_1 - 1} E_k(f)_p^{q_1}\right\}^{1/q_1} + \|f\|_{L_p(S_{d-1})}\right] \tag{10.3}$$

and

$$E_n(f)_q \leqslant C\left\{\sum_{k=n}^\infty k^{\theta q_1 - 1} E_k(f)_p^{q_1}\right\}^{1/q_1}, \tag{10.4}$$

where $q_1 = \begin{cases} q, & q < \infty \\ 1, & q = \infty \end{cases}$ *and* $\theta = (d-1)\left(\frac{1}{p} - \frac{1}{q}\right)$.

Proof. The Nikol'skii inequality for $0 < p \leqslant q \leqslant \infty$

$$\|P_n\|_{L_q(S_{d-1})} \leqslant C n^{(d-1)(\frac{1}{p}-\frac{1}{q})} \|P_n\|_{L_p(S_{d-1})}, \qquad P_n \in \operatorname{span}\left\{\bigcup_{k=1}^{n} H_k\right\} \tag{10.5}$$

was proved in Lemma 7.4 of [Be-Da-Di], and hence Theorem 4.1 implies our theorem. □

Actually (10.5) for $1 \leqslant p \leqslant q \leqslant \infty$ was proved by Kamzolov [Ka], and using Theorem 6.1 here, this implies (10.5).

The smoothness can be given by the K-functional

$$K_r\left(f, \widetilde{\Delta}, t^{2r}\right)_p = \inf_{g \in C^{2r}(S_{d-1})} \left(\|f - g\|_{L_p(S_{d-1})} + t^{2r} \|\widetilde{\Delta}^r g\|_{L_p(S_{d-1})}\right) \tag{10.6}$$

for $1 \leqslant p \leqslant \infty$. We can now state and prove an analogue of the Ul'yanov inequality.

Theorem 10.2. *For $f \in L_p(S_{d-1})$, $1 \leqslant p < q \leqslant \infty$ we have for integer $r \geqslant 1$*

$$\|f\|_{L_q(S_{d-1})} \leqslant C \left[\left\{\int_0^1 u^{-q_1\theta} K_r\left(f, \widetilde{\Delta}, u^{2r}\right)_p^{q_1} \frac{du}{u}\right\}^{1/q_1} + \|f\|_{L_p(S_{d-1})}\right] \tag{10.7}$$

and

$$K_r\left(f, \widetilde{\Delta}, t^{2r}\right)_q \leqslant C \left\{\int_0^t u^{-q_1\theta} K_r\left(f, \widetilde{\Delta}, u^{2r}\right)_p^{q_1} \frac{du}{u}\right\}^{1/q_1}, \tag{10.8}$$

where $q_1 = \begin{cases} q & q < \infty \\ 1 & q = \infty \end{cases}$, $\theta = (d-1)\left(\frac{1}{p} - \frac{1}{q}\right)$ *and* $K_r(f, \widetilde{\Delta}, t^{2r})$ *is given by* (10.6).

Proof. We set in Theorems 4.3 and 4.4

$$\Omega^{\gamma}(f, t)_p = \Omega^{2r}(f, t)_p = K_r\left(f, \widetilde{\Delta}, t^{2r}\right)_p.$$

We use $\Phi(g)_p = \|\widetilde{\Delta}^r g\|_{L_p(S^{d-1})}$. The Jackson-type theorem, which is the needed condition (4.9), was proved in [Ch-Di, Theorem 8.1, (8.8)], and hence we complete the proof of (10.7). The appropriate form of (4.11) with $P_n \in \operatorname{span}\left\{\bigcup_{k=0}^{n} H_k\right\}$ satisfying $\|P_n - f\|_{L_p(S_{d-1})} = E_n(f)_p$ is an immediate consequence of the definition of $K_r(f, \widetilde{\Delta}, t^{2r})_q$ as a K-functional. Since for $\varphi \in \operatorname{span}\left\{\bigcup_{k=0}^{n} H_k\right\}$, $\widetilde{\Delta}\varphi \in \operatorname{span}\left\{\bigcup_{k=0}^{n} H_k\right\}$ the necessary (4.12) is just (10.5). To establish (4.13) we use [Ch-Di, Theorem 8.2, (8.15)] with our notations. Therefore, (10.8) follows. □

We could have used in Theorem 10.2 one of the many moduli of smoothness in Rustamov [Ru] which are equivalent to the K-functional in (10.6), but they, like the K-functional, cannot be defined for $0 < p < 1$. A recently introduced set of moduli [Di,II] can be defined for $0 < p \leqslant \infty$, but while there are many results about it, we do not have the appropriate form of (4.9) and (4.13) for $0 < p < 1$, and hence using it would not improve the range of Theorem 10.2.

11. Jacobi weights

For the cube $I_d = [-1, 1] \times \cdots \times [-1, 1]$ the rate of best-weighted approximation is given by

$$E_n(f)_p = \inf \left(\|f - P_n\|_{L_{p,w_{\alpha,\beta}}(I^d)};\ P_n \in \Pi_n \right), \tag{11.1}$$

where Π_n is the class of polynomials of total degree $\leqslant n$,

$$w_{\alpha,\beta}(x) = \prod_{i=1}^{d} w_{\alpha_i,\beta_i}(x_i), \quad w_{\alpha_i,\beta_i}(x_i) = (1 - x_i)^{\alpha_i}(1 + x_i)^{\beta_i}$$

and $\alpha_i > -1$, $\beta_i > -1$. We have now the following corollary of Theorem 4.1.

Theorem 11.1. *Suppose* $f \in L_{p,w_{\alpha,\beta}}[I_d]$, $\alpha_i > -1$, $\beta_i > -1$, $\alpha_i + \beta_i > -1$ *and* $0 < p < q \leqslant \infty$. *Then*

$$\|f\|_{L_{q,w_{\alpha,\beta}}[I_d]} \leqslant C \left[\left\{ \sum_{k=1}^{\infty} k^{\theta q_1 - 1} E_k(f)_p^{q_1} \right\}^{1/q_1} + \|f\|_{L_{p,w_{\alpha,\beta}}[I_d]} \right] \tag{11.2}$$

and

$$E_n(f)_q \leqslant C \left\{ \sum_{k=n}^{\infty} k^{\theta q_1 - 1} E_k(f)_p^{q_1} \right\}^{1/q_1}, \tag{11.3}$$

where

$$q_1 = \begin{cases} q & q < \infty \\ 1 & q = \infty \end{cases} \quad \text{and } \theta = \left(\frac{1}{p} - \frac{1}{q} \right) \left(\sum_{i=1}^{d} \max(2 + 2 \max(\alpha_i, \beta_i), 1) \right).$$

Proof. Using the Nikol'skii-type inequality in Theorem 6.6 and Theorem 4.1, we obtain the present result. □

For the analogue of the Ul'yanov result here we have the one-dimensional case for $1 \leqslant p \leqslant \infty$. The K-functional is given by

$$K_r \left(f, P_{\alpha,\beta}(D), t^{2r} \right)_p = \inf \left(\|f - g\|_{L_p[-1,1]} + t^{2r} \|P_{\alpha,\beta}(D)^r g\|_{L_p[-1,1]} \right), \tag{11.4}$$

where $\alpha > -1$, $\beta > -1$ and

$$P_{\alpha,\beta}(D) = \frac{1}{(1-x)^{\alpha}(1+x)^{\beta}} \frac{d}{dx} \left((1 - x^2)(1 - x)^{\alpha}(1 + x)^{\beta} \right) \frac{d}{dx}. \tag{11.5}$$

Theorem 11.2. *Suppose* $f \in L_{p,w_{\alpha,\beta}}[-1,1]$, $\alpha+\beta > -1$, $\alpha > -1$, $\beta > -1$ *and* $1 \leqslant p < q \leqslant \infty$. *Then for any integer* $r \geqslant 1$

$$\|f\|_{L_{q,w_{\alpha,\beta}}[-1,1]} \leqslant C\left[\left\{\int_0^1 u^{-q_1\theta} K_r\left(f, P_{\alpha,\beta}(D), u^{2r}\right)_p^{q_1} \frac{du}{u}\right\}^{1/q_1} + \|f\|_{L_{p,w_{\alpha,\beta}}[-1,1]}\right] \tag{11.6}$$

and

$$K_r\left(f, P_{\alpha,\beta}(D), t^{2r}\right)_q \leqslant C\left\{\int_0^t u^{-q_1\theta} K_r\left(f, P_{\alpha,\beta}(D), u^{2r}\right)_p^{q_1} \frac{du}{u}\right\}^{1/q_1}, \tag{11.7}$$

where

$$q_1 = \begin{cases} q, & q < \infty \\ 1, & q = \infty \end{cases}, \quad \theta = \left(\frac{1}{p} - \frac{1}{q}\right) \max\left(2 + 2\max(\alpha,\beta), 1\right),$$

and $K_r\left(f, P_{\alpha,\beta}(D), t^{2r}\right)_p$ *is given by* (11.4).

Proof. We set in Theorems 4.3 and 4.4

$$\Omega^{\gamma}(f,t)_p = \Omega^{2r}(f,t)_p = K_r\left(f, P_{\alpha,\beta}(D), t^{2r}\right)_p.$$

We use here $\Phi(P_n)_p = \|P_{\alpha,\beta}(D)^r P_n\|_{L_p[-1,1]}$, $P_n \in \Pi_n$. The Jackson-type estimate, which is the needed inequality (4.9), was proved in [Ch-Di, (5.22)] for α and β as prescribed, and hence we have (11.6). The appropriate form of (4.11) is an immediate consequence of the definition of $K_r\left(f, P_{\alpha,\beta}(D), t^{2r}\right)_q$ as given (for q instead of p) by (11.4). The necessary inequality (4.12) is the Nikol'skii-type inequality (Theorem 6.6 for the special case dealt with here) since $P_{\alpha,\beta}(D)P_n \in \Pi_n$ if $P_n \in \Pi_n$. To establish (4.13) we note that it is in [Ch-Di, Theorem 5.6, A]. Therefore, we have (11.7). □

12. Concluding remarks

It is clear that many theorems in this paper could be extended if the ingredients in former papers were extended. In particular, this applies to Sections 7–11 to varying degrees. We would like also to conjecture a simple Ul'yanov-type result for which the methods of this paper do not seem to be appropriate.

Conjecture. *For a domain* $\Gamma \subset \boldsymbol{R}^d$ *satisfying some simple restrictions* (*say for instance* $\Gamma = \{x;\ |x| \leqslant 1\}$) *and* $0 < p < q \leqslant \infty$ *one has*

$$\|f\|_{L_q(\Gamma)} \leqslant C\left[\left\{\int_0^1 u^{-q_1\theta}\omega^r(f,u)_p^{q_1}\frac{du}{u}\right\}^{1/q_1} + \|f\|_{L_p(\Gamma)}\right] \tag{12.1}$$

and

$$\omega^r(f,t)_q \leqslant C \left\{ \int_0^t u^{-q_1\theta} \omega^r(f,u)_p^{q_1} \frac{du}{u} \right\}^{1/q_1}, \tag{12.2}$$

where

$$q_1 = \begin{cases} q & q < \infty \\ 1 & q = \infty \end{cases}, \quad \theta = \left(\frac{1}{p} - \frac{1}{q}\right) d \quad \textit{and} \quad \omega^r(f,t)_p = \sup_{|h| \leqslant t} \|\Delta_h^r f\|_{L_p(\Gamma)}.$$

While probably special cases of the above are known or easy to prove, we would applaud a result of the type that is valid for $p > 0$, $q_1 \neq 1$ and $d > 1$.

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